Topic 5:

Direct Proofs of $p \to q$

Handful O’ Definitions (1 / 2)

Definition: Conjecture

Definition: Theorem

Definition: Proof

Handful O' Definitions (2 / 2)

Definition: Lemma

Definition: Corollary

Example(s):

Why do People Fear Proofs?

1. Proofs don't come from an assembly line.
   - Need knowledge, persistence, and creativity
2. Creating proofs seems like magic.
   - But they are systematic in many ways
3. Proofs are hard to read and understand.
   - Only if the writer makes them so
4. Institutionalized Fear.
   - Many teachers avoid them in classes

Constructing a proof? Remember:

1. There are several proof techniques for a reason.
   - One may be easier to use than the others
2. Knowledge of mathematics is important.
   - Remember our Math Review?
3. There are “tricks” to know.
   - Ex: Dividing an even # in half leaves no remainder
4. Practice helps . . . a lot!
   - Just as it does for most everything else
5. Dead ends are expected.
   - Proofs in books are the final, polished versions

Speaking of Even Numbers . . .

Here are a few ways to say that $n \in \mathbb{Z}$ is even:

- $n/2$ is an integer
- $n \,\%\, 2 = 0$
- n is twice some integer (e.g., $n = 2k, k \in \mathbb{Z}$)

Similarly, $d \in \mathbb{Z}$ is odd when:

Types Of Proof In This Class

1. Direct Proof
   - The most common variety
2. Proof by Contraposition
   - Like Direct, but with a twist
3. Proof by Contradiction
   - A dark road on a foggy night

Direct Proofs

Our First Conjecture

Conjecture: **If n is even, then n^2 is also even, $n \in \mathbb{Z}$.**

Proof-Writing Miscellanea

- Remember: A conjecture isn't a theorem until proven.
- Don't lose sight of your destination.
- When writing proofs in this class:
  1. Always start with "Proof (*style*):"
  2. Stating your allowed assumptions can help.
  3. Define all introduced variables.
  4. End proofs with "Therefore, " and the conjecture.

     [Outside of this class: "Q.E.D." (*quod erat demonstrandum*, Latin for "this was to be demonstrated.")]

A Conjecture About Rational Numbers

Conjecture: **If $r, s \in \mathbb{Q}$, then $\frac{r}{s} \in \mathbb{Q}$.**

A Conjecture About Inequalities

Conjecture: **If $0 < a < b$, then $a^2 < b^2$, $a, b \in \mathbb{R}$.**

"Proof By Cases"

Question: **How would you prove that $\forall x\, C(x)$ is true, where $x \in \{6, 28, 496\}$?**

A Direct Proof Employing Cases

Conjecture: $s \to r \;\equiv\; \neg r \to \neg s$

Proof (direct): Consider all possible combinations of values of r and s:

	r	s	$s \to r$	$\neg r \to \neg s$
Case 1:	T	T	**T**	**T**
Case 2:	T	F	**T**	**T**
Case 3:	F	T	**F**	**F**
Case 4:	F	F	**T**	**T**

Therefore, $s \to r \;\equiv\; \neg r \to \neg s$.

(Yes, this truth table is a direct proof by cases.)

A More Traditional Direct Proof With Cases

Conjecture: $3x(x+1)$ **is even,** $x \in \mathbb{Z}$.

A More Interesting Direct Proof With Cases

Conjecture: $x^2 \% 4 \in \{0, 1\}$, $x \in \mathbb{Z}$.

Poor Arguments $\longrightarrow$ Poor Proofs (1 / 2)

Conjecture: $1 < 0$.

Proof or Goof?:

Consider x such that $0 < x < 1$. Take the base–10 logarithm of both sides of $x < 1$: $log_{10}x < log_{10}1$. By definition, $log_{10}1 = 0$. Divide both sides by $log_{10}x$: $\frac{log_{10}x}{log_{10}x} < \frac{0}{log_{10}x}$, which reduces to $1 < 0$.

Therefore, $1 < 0$.

Poor Arguments $\longrightarrow$ Poor Proofs (2 / 2)

Conjecture: For all $n \in \mathbb{Z}^{odd}$, $(n^2 - 1)\%4 = 0$.

Proof or Goof?:

Let $x = 1$. $1^2 - 1 = 0$. $0\%4 = 0$. Let $x = 3$. $3^2 - 1 = 8$. $8\%4 = 0$. Let $x = 5$. $5^2 - 1 = 24$. $24\%4 = 0$. This shows no sign of failing to give a result of 0.

Therefore, for all $n \in \mathbb{Z}^{odd}$, $(n^2 - 1)\%4 = 0$.

Disproving Conjectures (1 / 2)

There are two common approaches:

(1) Prove that the conjecture's negation is true.

Conjecture: If m is even and n is even, $m + n$ is odd.

Disproving Conjectures (2 / 2)

(2) Find a counter–example. (Very commonly used!)

Conjecture: No integer n exists such that the sum of its divisors equals $2n$.