Topic 02: Structural Induction
Structural Induction
(over lists)

So we have a linked list...
Structural Induction
(over lists)

If we can prove it for a list of length 1...
Structural Induction
(over lists)

...and also prove it for a long list, assuming it's true for a slightly shorter one...
Structural Induction
(over lists)

...then it's true for a list of any length.
This looks a lot like induction over the integers (and it should!).

But we can also think of this as induction over a **structure**. We started with a simple structure (base case), and then used induction to prove it for larger ones.
Structural Induction
(over trees)

So we have a tree...
Structural Induction
(over trees)

If we can prove the predicate for the leaves...
Structural Induction
(over trees)

...and also prove it for any arbitrary node (assuming that it holds for its children)...

[Tree diagram with light blue nodes representing the induction process]
Structural Induction
(over trees)

...then it holds for the entire tree.
Structural Induction Over Trees

There are three common strategies for performing induction over trees:

- root + 2 subtrees
- add one more leaf
- add a whole row of leaves
Structural Induction Over Trees

There are three common strategies for performing induction over trees:

- **root + 2 subtrees**
- add one more leaf
- add a whole row of leaves

Try this one **first**!

It often works better, and more easily, than the others.

It's not always the best, but it's worth trying.
Strategy 1: root + 2 subtrees

Assume (the I.H.) that the conjecture holds for trees of height \( h - 1 \)
Strategy 1: root + 2 subtrees

This means that it applies to both subtrees.
Strategy 1: The whole tree is thus 1 root node, plus 2 subtrees of size \( h - 1 \) (or less).
Strategy 1: root + 2 subtrees

Use this to (hopefully) prove that the conjecture holds for trees of height $h$. 
Strategy 1: root + 2 subtrees

Pitfalls:
Remember that either (or both) subtrees might be small – maybe even empty!
Strategy 2: add a leaf

Assume (the I.H.) that the conjecture holds for trees with $n - 1$ nodes.
Strategy 2: add a leaf

Add a single leaf, thus constructing a tree with $n$ nodes.
Strategy 2: add a leaf

Use this to (hopefully) prove that the conjecture holds for trees with $n$ nodes.
Strategy 2: add a leaf

Pitfalls:

Remember that the tree might not be complete.

It might have holes!
Strategy 3: add a row of leaves

Strategy 3:
Assume (the I.H.) that the conjecture holds for trees of height \( h - 1 \).
Strategy 3: add a row of leaves

Strategy 3:

Add a whole row of new leaves.

A complete row, at height $h$, has $2^h$ leaves.
Strategy 3: add a row of leaves

Use this to (hopefully) prove that the conjecture holds for trees of height $h - 1$. 
Strategy 3: add a row of leaves

Pitfalls:

It can be difficult to use this technique if you aren't allowed to assume that the tree is complete.
Conjecture:
In a non-empty, complete binary tree, the number of internal nodes is always one less than the number of leaves.

Try to prove this using **both** the “root+2 subtrees” and “add a row” strategies.
Structural Induction – Solution #1

- **Base Case:**

  Consider a tree with a single node (height 0). It has a single leaf, and no internal nodes; the conjecture is trivially true.

  Why didn't we consider an empty tree?

  (Sometimes, it's a **critical** base case!)
Structural Induction – Solution #1

• Inductive Case:

Assume that the conjecture holds for all trees of height $h - 1$ or less. We will prove that it holds for a tree of height $h$.

Consider a tree of height $h$. It is made up of a single root node, and two sub-trees, each with height exactly $h - 1$. 
Structural Induction – Solution #1

• Inductive Case (cont):

By the I.H., we know that both subtrees have exactly one fewer internal nodes than they have leaves. If each subtree has $L$ leaves, then the two subtrees, between them, have $2L$ leaves and $2L - 2$ internal nodes.
Structural Induction – Solution #1

- **Inductive Case (cont):**

  Since the root is also an internal node, the overall tree has $2^L$ leaves and $2^L - 1$ internal nodes.

  Therefore, the conjecture holds for a tree of height $h$.

- **Conclusion:**

  Therefore, the conjecture holds for all non-empty, complete trees.
Structural Induction – Solution #2

- **Base Case:**
  Consider a tree with a single node (height 0). It has a single leaf, and no internal nodes; the conjecture is trivially true.
Structural Induction – Solution #2

• Inductive Case:

Assume that the conjecture holds for all trees of height \( h - 1 \) or less. We will prove that it holds for a tree of height \( h \).

Consider a tree of height \( h \). By the I.H., it has exactly one more leaf than internal nodes; we will thus say that it has \( L \) leaves, and \( L - 1 \) internal nodes.
Structural Induction – Solution #2

• Inductive Case (cont):

Now, imagine that we add a new layer to the bottom of the tree. To do this, we take every leaf, and give that leaf two children, each of which are new leaves. Each old leaf becomes an internal node.

Since there were $L$ leaves in the smaller tree, there are $2L$ leaves in the new tree, and $L$ nodes have switched from being leaves to internal nodes.
• **Inductive Case (cont):**

Thus, the total number of internal nodes in the new tree is $2^L - 1$, and the total number of leaves in the new tree is $2^L$.

Thus, the conjecture holds for a complete tree of height $h$.

• **Conclusion:**

Thus, the conjecture holds for all non-empty, complete trees.
Another Exercise...

The conjecture also has a direct proof.

Now find that.
Direct Proof - Solution

• Direct Proof:

In a complete binary tree, every level $k$ of the tree has exactly $2^k$ nodes. Thus, a complete binary tree of height $h$ has exactly $2^0 + 2^1 + 2^2 + \ldots + 2^h = 2^{h+1} - 1$ nodes.

Of those nodes, exactly $2^h$ are leaves, meaning that $2^h - 1$ of the nodes are internal nodes.

Thus, for all non-empty, complete binary trees, there is exactly one fewer internal node than leaves.
Induction Failures

The next few slides have examples of common induction errors I've seen. Can you spot the problem?

(We'll use the previous conjecture as the example problem.)
Induction Failures (#1)

Base Case:
A complete tree of height 1 has three nodes: the root, and two children. It has one internal node, and two leaf nodes; the conjecture holds trivially.
Induction

Base Case:

A complete tree of height 1 has three nodes: the root, and two children. It has one internal node, and two leaf nodes; the conjecture holds trivially.

ERROR: Wrong base

The proof does not consider a tree of height 0 (a single node).
Induction Failures (#2)

Base Case:
...tree of height 0...

Inductive Case:
Consider a tree of height 1. It is made up of a single root node, and two subtrees of height 0. It has two leaves, and an internal node (the root); thus, the conjecture holds.
Induction Failures (#2)

**Base Case:**
...tree of height 0...

**Inductive Case:**
Consider a tree of height 1. It is made up of a single root node, and two subtrees of height 0. It has two leaves, and an internal node (the root); thus, the conjecture holds.

**ERROR:** Not general
The “inductive” step is nothing more than a 2\textsuperscript{nd} base case. It does not prove that the conjecture holds for all situations.
Induction Failures (#3)

Inductive Case:

Consider a tree of height $h - 1$. Assume that it has $N$ internal nodes, and $L$ leaves. By the I.H., $N = L - 1$.

Each node can have two children, so in a tree of height $h$, we have $2N$ internal nodes and $2L - 2$ leaves. Thus ...
**Inductive Case:**

Consider a tree of height $h$. Assume that it has $N$ internal nodes and $L$ leaves. By the I.H., $N = L - 1$.

Each node can have two children, so in a tree of height $h$, we have $2N$ internal nodes and $2L - 2$ leaves. Thus ...

**ERROR:** Fallacious reasoning

It is true that every internal node has exactly two children in a complete binary tree.

But this doesn't mean that there are 2 times as many internal nodes when we add a new layer.
Inductive Case:

Consider a complete tree of height $h - 1$. Assume that it has $N$ internal nodes, and $L$ leaves. By the I.H., $N = L - 1$.

Each level of the tree, of height $i$, has $2^i$ nodes; thus, the entire tree of height $h - 1$ has $2^0 + 2^1 + 2^2 + \ldots + 2^{h-1} = 2^h - 1$ nodes.

However, a tree of height $h$ has $2^h$ leaves. This is one more than the internal nodes; thus, the conjecture holds.
Inductive Case:

Consider a complete tree of height \( h - 1 \). Assume that it has \( N \) internal nodes, and \( L \) leaves. By the I.H., \( N = L - 1 \).

Each level of the tree, of height \( i \), has \( 2^i \) nodes; thus, the entire tree of height \( h - 1 \) has \( 2^0 + 2^1 + 2^2 + \ldots + 2^{h-1} = 2^h - 1 \) nodes.

However, a tree of height \( h \) has \( 2^h \) leaves. This is one more than the internal nodes; thus, the conjecture holds.

**ERROR:** Not Inductive

This proof is actually a disguised version of the Direct Proof.
Summary:

- Structural induction is just another form of induction
  - Base: Something simple
  - Inductive: More complex
- Be clear, be precise!