Homework 2 Solutions

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Problem 1

```
min = max = list[0];                o+=4
                                    o++
for(int i = 1; i < n-1; i+=2){
                                  o+=2
                                  o+=6
    if(list[i] < list[i+1]){
        if(list[i] < min)         o+=3
            min = list[i];        o+=3
        if(list[i+1] > max)       o+=4
            max = list[i+1];      o+=4
    } else {
        if(list[i+1] < min)       o+=4
            min = list[i+1];      o+=4
        if(list[i] > max)         o+=3
            max = list[i];        o+=3
    }
                                  o+=2
}
                                  o+=2

if(i == n-1){                     o+=2
    if(list[i] < min)             o+=3
        min = list[i];            o+=3
    if(list[i] > max)             o+=3
        max = list[i];            o+=3
}
```

Total: 24(n-2/2) + +21
= 12n - 24 + 21
= 12n - 3

Problem 2

$$2 < log_3 n < log_2 n < n^{\frac{2}{3}} < 20n < 4n^2 < 3^n < n! \quad (1)$$

Problem 3

To prove that $f(n) \in O(n)$, show $f(n) \leq c(g(n)) \forall n_0 >= n$
To prove that $f(n) \in \Omega(n)$, show $f(n) \geq c(g(n)) \forall n_0 >= n$

1. $c_1 n$
   (a) $O(n)$: Choose a c such that $c > c_1$
   (b) $\Omega(n)$: Choose a c such that $c < c_1$
2. $c_2 n^3 + c_3$

(a) $O(n)$: we know that by the polynomial theorem that $c_2n^3 + c_3 = O(n^3)$

(b) $\Omega(n)$: We want $c_2n^3 + c_3 \geq cn^3$, so solve $\frac{c_2n^3+c_3}{n^3} \geq c$, $c_2 + \frac{c_3}{n^3} \geq c$. Choosing n_0 to be any number, say 1, gives $c_2 \geq c$ (this is true because $\frac{c_3}{n^3}$ becomes smaller as n increases0. We apply similar logic for the rest of the problems.

3. $c_4nlogn + c_5n$

(a) $O(n)$: $c_4nlogn + c_5n \leq cnlogn$, $\frac{c_4nlogn+c_5n}{nlogn} \leq c$, $c_4 + \frac{c_5}{logn} \leq c$. Choose $n_0 = 2$ and a $c_4 + c_5 \leq c$

(b) $\Omega(n)$: $c_4nlogn + c_5n \geq cnlogn$, $\frac{c_4nlogn+c_5n}{nlogn} \geq c$, $c_4 + \frac{c_5}{logn} \geq c$. Choose $n_0 = 2$ and a $c_4 \geq c$

4. $c_62^n + c_7n^6$

(a) $O(n)$: $c_62^n + c_7n^6 \leq c2^n$, $\frac{c_62^n+c_7n^6}{2^n} \leq c$, $c_6 + \frac{c_7n^6}{2^n} \leq c$. Using l'Hopital's rule, we find that $\lim_{n\to\infty} \frac{c_7n^6}{2^n} = 0$, which says that 2^n grows faster than n^6. So, just find an n_0 such that $n^6 \leq 2^n$. $n_0 = 30 \to 30^6 \leq 2^{30} \to 729000000 \leq 1073741824$. Now just pick $c \geq c_6 + c_7$

(b) $\Omega(n)$: $c_62^n + c_7n^6 \geq c2^n$, $c_6 + \frac{c_7n^6}{2^n} \geq c$. Pick $c \leq c_6 + c_7$. We know from the our Big-O calculations that $\frac{c_7n^6}{2^n} < 1$ after $n_0 = 30$.

Problem 4

1. (c) $O(n^2)$
2. (e) $O(nlogn)$
3. (g) $O(n^2logn)$
4. (i) $O(n)$

Problem 5

We can use the Master Theorem because $a \geq 1$, $b > 1$, $c > 0$, and $k \geq 0$. Using the Master Theorem: $a = 1$, $b = 2$, $c = 1$, $k = \frac{1}{2}$, so $1 < \sqrt{2}$, so this is $O(\sqrt{n})$

Problem 6

$$T(n) = 2T(\frac{n}{2}) + n, T(2) = 2$$

$$T(\frac{n}{2}) = 2T(\frac{n}{2^2}) + \frac{n}{2}$$

$$T(\frac{n}{2^2}) = 2T(\frac{n}{2^3}) + \frac{n}{2^2}$$

$$T(\frac{n}{2}) = 2(2T(\frac{n}{2^3}) + \frac{n}{2^2}) + \frac{n}{2}$$

$$T(\frac{n}{2}) = 2^2T(\frac{n}{2^3}) + 2\frac{n}{2^2} + \frac{n}{2}$$

$$T(n) = 2(2^2T(\frac{n}{2^3}) + 2\frac{n}{2^2} + \frac{n}{2}) + n$$

$$T(n) = 2^3T(\frac{n}{2^3}) + 3n$$

$$T(n) = 2^kT(\frac{n}{2^k}) + kn$$

Setting $\frac{n}{2^k} = 2$, $n = 2^{k+1}$, $log_2n = k + 1$, $log_2n - 1 = k$

$$T(n) = 2^{log_2n-1}2 + (log_2n - 1)n$$

$$T(n) = n + nlog_2n - n$$

$$T(n) = nlog_2 n$$

According to Master Theorem, $T(n) = O(nlogn)$

Proof (Inductive):

Basis: n=2

$T(2) = 2$ by definition. And our closed form says $nlogn = 2 * log_2 2 = 2$

Inductive: $T(n) = nlogn \rightarrow T(2n) = (2nlog(2n))$

$T(2n) = 2T(n) + n$

From IH, $T(2n) = 2nlogn + 2n$

$T(2n) = 2n(logn + 1)$

$T(2n) = 2nlog2n$

QED