2-d Collisions

Based on “2-Dimensional Elastic Collisions Without Trigonometry” by Chad Bercheck.
http://www.geocities.com/vobarian/2dcollisions/

Known:
Particle 1:
- mass: m_1
- velocity: $\mathbf{v_{1x}}, \mathbf{v_{1y}}$
- position: x_1, y_1

Particle 2:
- mass: m_2
- velocity: $\mathbf{v_{2x}}, \mathbf{v_{2y}}$
- position: x_2, y_2

Need, where f = final = after the collision:
Particle 1:
- velocity: $\mathbf{v_{1xf}}, \mathbf{v_{1yf}}$

Particle 2:
- velocity: $\mathbf{v_{2xf}}, \mathbf{v_{2yf}}$

Note about mass: If $m_1 == m_2$, then equations do get simpler. I show below where this assumption comes into the equations. The final result given here does assume that $m_1 == m_2$.

Find unit normal and unit tangent vectors:
n = normal vector
un = unit normal vector
- $\mathbf{un_x}$ = x-component of unit normal vector
- $\mathbf{un_y}$ = y-component of unit normal vector

t = tangent vector
ut = unit tangent vector
- $\mathbf{ut_x}$ = x-component of unit tangent vector
- $\mathbf{ut_y}$ = y-component of unit tangent vector

$\mathbf{n} = <x_2 - x_1, y_2 - y_1>$
- $\mathbf{un} = < (x_2 - x_1) / \text{sqrt}[], (y_2 - y_1) / \text{sqrt}[] >$
- $\mathbf{un_x} = (x_2 - x_1) / \text{sqrt}[]$
- $\mathbf{un_y} = (y2 - y1) / \text{sqrt}[]$

$\mathbf{t} = <-(y_2 - y_1), (x_2 - x_1) >$
- $\mathbf{ut} = < -(y_2 - y_1) / \text{sqrt}[], (x_2 - x_1) / \text{sqrt}[] >$
- $\mathbf{ut_x} = -(y_2 - y_1) / \text{sqrt}[]$
- $\mathbf{ut_y} = (x_2 - x_1) / \text{sqrt}[]$

Note: since $\text{sqrt}[(x_2 - x_1)^2 + (y_2 - y_1)^2]$ keeps appearing in the formulas, I am abbreviating it by using: sqrt[].

Find normal and tangent for $\mathbf{v_1}$ and $\mathbf{v_2}$:
For each of the two particles, find the magnitude of their velocities relative to the unit normal and unit tangent directions. This is done by taking a dot product of $\mathbf{v_1} \bullet \mathbf{n}$ and $\mathbf{v_1} \bullet \mathbf{t}$ (and analogously for $\mathbf{v_2}$). Since we know the x- and y-components of $\mathbf{v_1}$ and $\mathbf{v_2}$, we compute these as follows:

$v_{1n} = \mathbf{un} \bullet \mathbf{v_1} = un_x v_{1x} + un_y v_{1y} = [v_{1x}(x_2 - x_1) + v_{1y}(y_2 - y_1)] / \mathrm{sqrt}[]$
$v_{2n} = \mathbf{un} \bullet \mathbf{v_2} = un_x v_{2x} + un_y v_{2y} = [v_{2x}(x_2 - x_1) + v_{2y}(y_2 - y_1)] / \mathrm{sqrt}[]$
$v_{1t} = \mathbf{ut} \bullet \mathbf{v_1} = ut_x v_{1x} + ut_y v_{1y} = [-v_{1x}(y_2 - y_1) + v_{1y}(x_2 - x_1)] / \mathrm{sqrt}[]$
$v_{2t} = \mathbf{ut} \bullet \mathbf{v_2} = ut_x v_{2x} + ut_y v_{2y} = [-v_{2x}(y_2 - y_1) + v_{2y}(x_2 - x_1)] / \mathrm{sqrt}[]$
Note: v_{1n}, v_{1t}, v_{2n}, and v_{2t} are scalar values, not vectors. They can become vectors if we include their directions. For v_{1n} and v_{2n}, the direction is either in the same direction as, or the opposite direction of, the unit normal vector, **un**. For v_{1t} and v_{2t}, the direction is either in the same direction as, or the opposite direction of, the unit tangent vector, **ut**.

Find normal and tangent after collision for $\mathbf{v_1}$ and $\mathbf{v_2}$:
Find the normal and tangent values for both particles after the collision. This will give us $\mathbf{v_{1nf}}$, $\mathbf{v_{1tf}}$, $\mathbf{v_{2nf}}$ and $\mathbf{v_{2tf}}$, where f indicates the final velocity after the collision. The tangential velocities after the collision are the same for $\mathbf{v_{1f}}$ and $\mathbf{v_{2f}}$ as before the collision; this result derives from the conservation of momentum. The normal velocities of $\mathbf{v_{1f}}$ and $\mathbf{v_{2f}}$ are different after the collision; this result derives from the 1d collision and uses conservation of momentum and energy.
$\mathbf{v_{1tf} = v_{1t}}$
$\mathbf{v_{2tf} = v_{2t}}$
$\mathbf{v_{1nf}} = [v_{1n}(m_1 - m_2) + 2mv_{2n}] / (m_1 + m_2)$
If $m_1 == m_2$, this equation becomes:
$\mathbf{v_{1nf}} = [v_{1n}(m - m) + 2mv_{2n}] / (m + m) = 2mv_{2n} / (2m)$
$\mathbf{v_{1nf} = v_{2n}}$
$\mathbf{v2nf} = [v_{2n}(m_1 - m_2) + 2m_2 v_{1n}] / (m_1 + m_2)$
If $m_1 == m_2$, this equation becomes:
$v_{2nf} = [v_{2n}(m - m) + 2mv_{1n}] / (m + m) = 2mv_{1n} / (2m)$
$\mathbf{v_{2nf} = v_{1n}}$
From this point forward, assume $m_1 == m_2$.

Find $\mathbf{v_{1f}}$ and $\mathbf{v_{2f}}$:

For particle 1, we know: $\mathbf{v_{1nf}}$ and $\mathbf{v_{1tf}}$. We can use these two to find:

$$\mathbf{v_{1f}} = \mathbf{v_{1nf}} + \mathbf{v_{1tf}}$$

as shown in the drawing.

But, we need $\mathbf{v_{1xf}}$ and $\mathbf{v_{1yf}}$; that is, we need the x- and y-components of $\mathbf{v_{1f}}$. We compute:

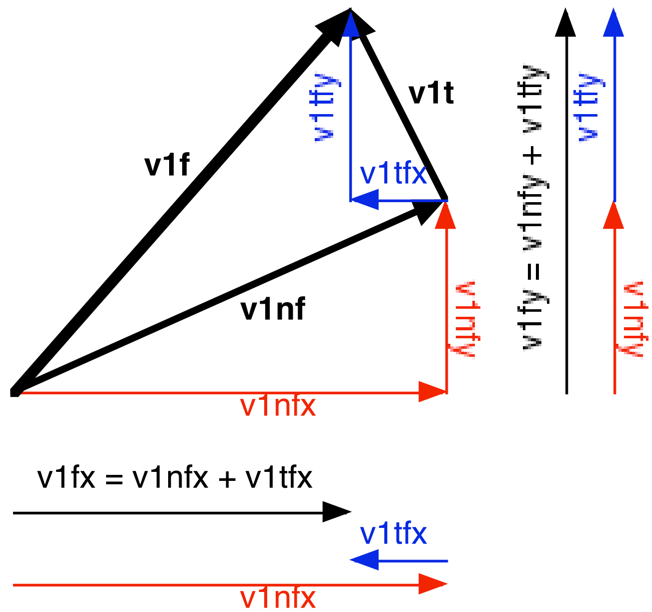
$$\mathbf{v_{1nf}} = \langle v_{1nfx}, v_{1nfy} \rangle$$

and

$$\mathbf{v_{1tf}} = \langle v_{1tfx}, v_{1tfy} \rangle$$

Then, we can add vectors to get:

$$\mathbf{v_{1f}} = \langle v_{1nfx} + v_{1tfx}, v_{1nfy} + v_{1tfy} \rangle.$$



First, find $\mathbf{v_{1nfx}}$. This is done by multiplying $\mathbf{v_{1nf}}$ by the x-component of the unit normal vector:

$\mathbf{v_{1nfx}} = \mathbf{v_{1nf}}((x_2 - x_1) / \text{sqrt}[]) = \mathbf{v_{2n}}((x_2 - x_1) / \text{sqrt}[]) =$

$([v_{2x} (x_2 - x_1) + v_{2y}(y_2 - y_1)] / \text{sqrt}[]) ((x_2 - x_1) / \text{sqrt}[]) =$

$[v_{2x} (x_2 - x_1)^2 + v_{2y}(x_2 - x_1)(y_2 - y_1)] / [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

Now, find $\mathbf{v_{1tfx}}$. This is done by multiplying $\mathbf{v_{1tf}}$ by the x-component of the unit tangent vector:

$\mathbf{v_{1tfx}} = \mathbf{v_{1tf}}((x_2 - x_1)/ \text{sqrt}[]) = \mathbf{v_{1t}}((x_2 - x_1) / \text{sqrt}[]) =$

$([-v_{1x}(y_2 - y_1) + v_{1y}(x_2 - x_1)] / \text{sqrt}[]) ((y_2 - y_1) / \text{sqrt}[]) =$

$[-v_{1x} (x_2 - x_1)(y_2 - y_1) + v_{1y} (x_2 - x_1)^2] / [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

Finally, to find $\mathbf{v_{1xf}}$, add $\mathbf{v_{1nfx}} + \mathbf{v_{1tfx}}$. Note that both terms have the same denominator.

$\mathbf{v_{1fx}} = \mathbf{v_{1nfx}} + \mathbf{v_{1tfx}} =$

$\{[v_{2x}(x_2 - x_1)^2 + v_{2y}(x_2 - x_1)(y_2 - y_1)] + [-v_{1x}(x_2 - x_1)(y_2 - y_1) + v_{1y}(x_2 - x_1)^2]\}$

$/ [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

Repeat the last three steps for $\mathbf{v_2}$:

First, find $\mathbf{v_{2nfx}}$. This is done by multiplying $\mathbf{v_{2nf}}$ by the x-component of the unit normal vector:

$\mathbf{v_{2nfx}} = \mathbf{v_{2nf}}((x_2 - x_1) / \text{sqrt}[]) = \mathbf{v_{1n}}((x_2 - x_1) / \text{sqrt}[]) =$

$([v_{1x}(x_2 - x_1) + v_{1y}(y_2 - y_1)] / \text{sqrt}[]) ((x_2 - x_1) / \text{sqrt}[]) =$

$[v_{1x}(x_2 - x_1)^2 + v_{1y}(x_2 - x_1)(y_2 - y_1)] / [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

Now, find $\mathbf{v_{2tfx}}$. This is done by multiplying $\mathbf{v_{2tf}}$ by the x-component of the unit tangent vector:

$\mathbf{v_{2tfx}} = \mathbf{v_{2tf}}((x_2 - x_1) / \text{sqrt}[]) = \mathbf{v_{2t}}((x_2 - x_1) / \text{sqrt}[]) =$

$([-v_{2x}(y_2 - y_1) + v_{2y}(x_2 - x_1)] / \text{sqrt}[]) ((x_2 - x_1) / \text{sqrt}[]) =$

$[-v_{2x}(x_2 - x_1)(y_2 - y_1) + v_{2y}(x_2 - x_1)^2] / [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

Finally, to find $\mathbf{v_{2fx}}$, add $\mathbf{v_{2nfx}} + \mathbf{v_{2tfx}}$. Note that both terms have the same denominator.

$\mathbf{v_{2fx}} = \mathbf{v_{2nfx}} + \mathbf{v_{2tfx}} =$

$\{[v_{1x}(x_2 - x_1)^2 + v_{1y}(x_2 - x_1)(y_2 - y_1)] + [-v_{2x}(x_2 - x_1)(y_2 - y_1) + v_{2y}(x_2 - x_1)^2]\}$

$/ [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

To find the y-components for particles 1 and 2, we perform the same 3 steps.
First, find $\mathbf{v_{1nfy}}$. This is done by multiplying $\mathbf{v_{1nf}}$ by the y-component of the unit normal vector:

$\mathbf{v_{1nfy}} = \mathbf{v_{1nf}}((y_2 - y_1) / \text{sqrt}[]) = \mathbf{v_{2n}}((y_2 - y_1) / \text{sqrt}[]) =$
$([v_{2x}(x_2 - x_1) + v_{2y}(y_2 - y_1)] / \text{sqrt}[])\ ((y_2 - y_1) / \text{sqrt}[]) =$
$[v_{2x}(x_2 - x_1)(y_2 - y_1) + v_{2y}(y_2 - y_1)^2] / [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

Now, find $\mathbf{v_{1tfy}}$. This is done by multiplying $\mathbf{v_{1tf}}$ by the y-component of the unit tangent vector:

$\mathbf{v_{1tfy}} = \mathbf{v_{1tf}}((y_2 - y_1) / \text{sqrt}[]) = \mathbf{v_{1t}}((y_2 - y_1) / \text{sqrt}[]) =$
$([-v_{1x}(y_2 - y_1) + v_{1y}(x_2 - x_1)] / \text{sqrt}[])\ ((y_2 - y_1) / \text{sqrt}[]) =$
$[-v_{1x}(y_2 - y_1)^2 + v_{1y}(x_2 - x_1)(y_2 - y_1)] / [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

Finally, to find $\mathbf{v_{1yf}}$, add $\mathbf{v_{1nfy} + v_{1tfy}}$. Note that both terms have the same denominator.

$\mathbf{v_{1fy} = v_{1nfy} + v_{1tfy}} =$
$\{[v_{2x}(x_2 - x_1)(y_2 - y_1) + v_{2y}(y_2 - y_1)^2] + [-v_{1x}(y_2 - y_1)^2 + v_{1y}(x_2 - x_1)(y_2 - y_1)]\}$
$/ [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

Repeat the last three steps for $\mathbf{v_2}$:
First, find $\mathbf{v_{2nfy}}$. This is done by multiplying $\mathbf{v_{2nf}}$ by the y-component of the unit normal vector:

$\mathbf{v_{2nfy}} = \mathbf{v_{2nf}}((y_2 - y_1) / \text{sqrt}[]) = \mathbf{v_{1n}}((y_2 - y_1) / \text{sqrt}[]) =$
$([v_{1x}(x_2 - x_1) + v_{1y}(y_2 - y_1)] / \text{sqrt}[])\ ((y_2 - y_1) / \text{sqrt}[]) =$
$[v_{1x}(x_2 - x_1)(y_2 - y_1) + v_{1y}(y_2 - y_1)^2] / [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

Now, find $\mathbf{v_{2tfy}}$. This is done by multiplying $\mathbf{v_{2tf}}$ by the y-component of the unit tangent vector:

$\mathbf{v_{2tfy}} = \mathbf{v_{2tf}}((y_2 - y_1) / \text{sqrt}[]) = \mathbf{v_{2t}}((y_2 - y_1) / \text{sqrt}[]) =$
$([-v_{2x}(y_2 - y_1) + v_{2y}(x_2 - x_1)] / \text{sqrt}[])\ ((y_2 - y_1) / \text{sqrt}[]) =$
$[-v_{2x}(y_2 - y_1)^2 + v_{2y}(x_2 - x_1)(y_2 - y_1)] / [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

Finally, to find $\mathbf{v_{2fy}}$, add $\mathbf{v_{2nfy} + v_{2tfy}}$. Note that both terms have the same denominator.

$\mathbf{v_{2fy} = v_{2nfy} + v_{2tfy}} =$
$\{[v_{1x}(x_2 - x_1)(y_2 - y_1) + v_{1y}(y_2 - y_1)^2] + [-v_{2x}(y_2 - y_1)^2 + v_{2y}(x_2 - x_1)(y_2 - y_1)]\}$
$/ [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

Summary:

$\mathbf{v_{1fx}} = \{[v_{2x}(x_2 - x_1)^2 + v_{2y}(x_2 - x_1)(y_2 - y_1)] + [-v_{1x}(x_2 - x_1)(y_2 - y_1) + v_{1y}(x_2 - x_1)^2]\}$
$/ [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

$\mathbf{v_{1fy}} = \{[v_{2x}(x_2 - x_1)(y_2 - y_1) + v_{2y}(y_2 - y_1)^2] + [-v_{1x}(y_2 - y_1)^2 + v_{1y}(x_2 - x_1)(y_2 - y_1)]\}$
$/ [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

$\mathbf{v_{2fx}} = \{[v_{1x}(x_2 - x_1)^2 + v_{1y}(x_2 - x_1)(y_2 - y_1)] + [-v_{2x}(x_2 - x_1)(y_2 - y_1) + v_{2y}(x_2 - x_1)^2]\}$
$/ [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

$\mathbf{v_{2fy}} = \{[v_{1x}(x_2 - x_1)(y_2 - y_1) + v_{1y}(y_2 - y_1)^2] + [-v_{2x}(y_2 - y_1)^2 + v_{2y}(x_2 - x_1)(y_2 - y_1)]\}$
$/ [(x_2 - x_1)^2 + (y_2 - y_1)^2]$