Computational Geometry

Chapter 4

Linear Programming

On the Agenda

- Linear programming
- Duality
- Smallest enclosing disk

Linear Programming - Example

- Define:
  - i – types of foods ($1 \le i \le d$).
  - j – types of vitamins ($1 \le j \le n$).
  - x_i – the amount of food of type i.
  - a_{ji} – the amount of vitamin j in one unit of food i.
  - c_i – the number of calories in one unit of food i.
  - b_j – minimal required amount of vitamin j.
- Constraints (we need to consume some minimal amount of each vitamin):

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d \ge b_1$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nd}x_d \ge b_n$$

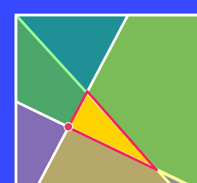
Minimize: $c^T x$
Subject to: $Ax \ge b$

- Minimize: the total number of calories consumed:

$$C(x) = c_1x_1 + c_2x_2 + \cdots + c_dx_d$$

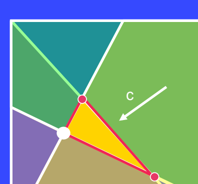
Linear Programming – The Geometry

- Each constraint defines defines a half-space region in d-dimensional space.
- The *feasible region* is the (convex) intersection of these half-spaces.
- We will treat the case $d = 2$, where each constraint defines a *half-plane*.



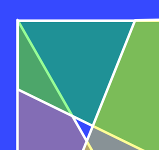
More Geometry

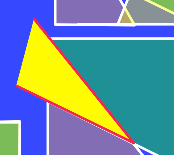
- The solution to the linear program is a point in the feasible region that is extreme in the direction of the target function.
- **Theorem:** Any bounded linear program that is feasible has a unique solution, which is a *vertex* of the feasible region.
- **Proof:** Convexity …

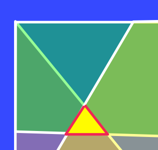


Degenerate Cases

- The feasible region may be:
  - Empty
  - Unbounded
- The solution may be:
  - Not unique



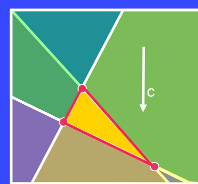




The Simplex Algorithm

- Assume WLOG that the cost function points "downwards".
- Construct (some of) the vertices of the feasible region.
- Walk edge by edge downwards until reaching a local minimum (which is also a global minimum).

- In R^d, the number of vertices might be $\Theta(n^{\lfloor d/2 \rfloor})$.



LP History

- Mid 20th century: Simplex algorithm, time complexity $\Theta(n^{\lfloor d/2 \rfloor})$ in the **worst** case.
- 1980's (Khachiyan) ellipsoid algorithm with time complexity poly(n,d).
- 1980's (Karmakar) interior-point algorithm with time complexity poly(n,d).
- 1984 (Megiddo) – parametric search algorithm with time complexity $O(C_d n)$ where C_d is a constant dependent only on d. E.g. $C_d = 2^{d^2}$.
- The holy grail: An algorithm with complexity independent of d.

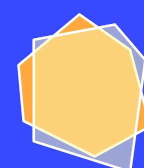
- In practice the simplex algorithm is used because of its linear *expected* runtime.

O(n log$n$) 2D Linear Programming

- Input:
  - n half planes.
  - Cost function that WLOG "points down".
- Algorithm:
  1. Partition the n half-planes into two groups.
  2. Compute, recursively, the feasible region for each group.
  3. Compute the intersection of the two feasible regions.
  4. Check the cost function on the region vertices.

Divide and Conquer – Complexity Analysis

- Stage 3:
  - Intersection of two convex polygons – plane sweep algorithm.
  - No more than four segments are ever in the SLS and no more than eight events in the EQ – $O(n)$.
- Stage 4:
  - Find the minimal cost vertex - $O(n)$.



$$T(n) = 2T(n/2)+O(n) \Rightarrow$$
$$T(n) = O(n \log n)$$

O(n^2) Incremental Algorithm

- The idea:
  - Start by intersecting two halfplanes.
  - Add halfplanes one by one and update optimal vertex by solving one-dimensional LP problem on new line *if needed*.

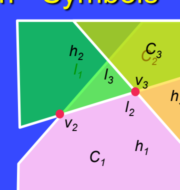
Incremental Algorithm - Symbols

h_i the i^{th} half plane

l_i the line that defines h_i

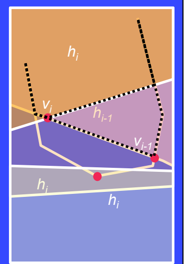
C_i the feasible region after i constraints

v_i the optimal vertex of C_i



Incremental Algorithm
Basic Theorem

- Theorem:
  1. if $v_{i-1} \in h_i$, then $v_i = v_{i-1}$. // O(1) check, nothing to do
  2. if $v_{i-1} \notin h_i$, then either
     $C_i = \emptyset$ // terminate
     or
     $C_i = C_{i-1} \cap h_i$ and v_i lies on l_i // run 1D LP
- Proof:
  1. Trivial. Otherwise v_i would not have been optimum before.

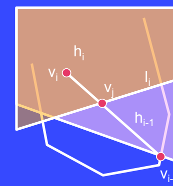


Basic Theorem - Cont.

2. Assume that v_i is not on l_i. v_i must be in C_{i-1}. By convexity, also the line $v_i v_{i-1}$ is in C_{i-1}.

   Consider point v_j - the intersection of $v_i v_{i-1}$ with l_i. v_j is in both C_{i-1} and C_i, and is better than v_i.

   Contradiction.

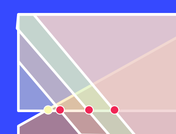


Finding v_i given l_i
(one-dimensional LP)

- Intersect each h_j ($j<i$) with l_i, generating i-1 rays representing (unbounded) intervals.
- Intersect the i-1 intervals in O(i) time.
- If the intersection is empty then report no solution, else report the lowest point.

Complexity Analysis

$$T(n) = \sum_{i=3}^{n} O(i) = O(n^2)$$



Incremental Algorithm – O(n)
Randomized Version

- Exactly like the deterministic version, only the order of the lines is random.
- **Theorem:** The expected runtime of the random incremental algorithm (over all n! permutations of the input constraints) is O(n).

Complexity Analysis

- The expected runtime is:

$$\sum_{i=3}^{n}\left[O(1)(1-E(x_i))+O(i)E(x_i)\right] \le O(n)+\sum_{i=3}^{n}\left[O(i)E(x_i)\right]$$

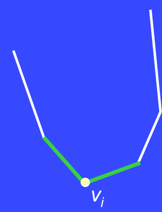
where x_i is a random variable:

$$x_i = \begin{cases} 1 & v_i \neq v_{i-1} \quad \text{// run 1D LP} \\ 0 & v_i = v_{i-1} \quad \text{// do nothing} \end{cases}$$

Probability Analysis

Backward analysis

- **Question**: When given a solution after *i* half-planes, what is the probability that the *last* half-plane affected the solution ?
- **Answer**: Exactly $2/i$, because a change can occur only if the last halfplane inserted is one of the two halfplanes thru v_i.
  (note that v_i depends on the *i* half-planes, but not on their order)



Complexity Analysis

$$E(x_i) = \Pr(v_i \neq v_{i-1}) \approx \frac{2}{i}$$

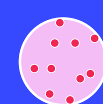
$$O(n) + \sum_{i=3}^{n} O(i)E(x_i) = O(n) + O\left(\sum_{i=3}^{n} i \cdot \frac{2}{i}\right) = O(n)$$

Just to Make Sure …

- False Claim:
  - The probabilistic analysis is for the average input. Hence there exist bad sets of constraints for which the algorithm's expected runtime is *more* than O(n), and there exist good sets of constraints for which the algorithm's expected runtime is *less* than O(n).
- True Claim:
  - The probabilistic analysis is valid for *all* inputs. The expected complexity is over all *permutations* of this input.

Smallest Enclosing Disk

- **Input:** *n* points.
- **Output**: Disk with minimal radius that contains all the points.



- **Theorem:** For any finite set of points in general position, the smallest enclosing disk either has at least three points on its boundary, or two points which form a diameter. If there are three points, they subdivide the circle into three arcs of length no more than π each. **Prove !**
- This immediately implies a O(n^4) algorithm (why ?).

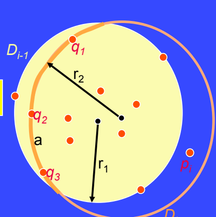
Basic Theorem

Theorem: Using an incremental algorithm, where D_i is the updated disk after seeing the first *i* points p_1 ,..., p_i:
If $p_i \notin D_{i-1}$ then p_i is on the *boundary* of D_i.

Proof:

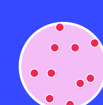
Observation: If $r_1 < r_2$ then $a < \pi$.

- $p_i \notin D_{i-1} \Rightarrow r_1 < r_2 \Rightarrow a < \pi$
- $p_i \notin \partial D_{i-1} \Rightarrow q_1, q_2, q_3 \in D_{i-1}$
  $\Rightarrow \mathrm{Arc}(q_1,q_3) > \pi$. Contradiction.



Incremental O(n) Expected Time Algorithm

- Construct the procedures:
  - MinDisk(P) – find a smallest enclosing disk for a set of points P.
  - MinDisk1(P,q) – find an enclosing disk for a set of points P which touches point q.
  - MinDisk2(P,q_1,q_2) – find an enclosing disk for a set of points P, which touches points q_1 and q_2.
  - Disk(q_1,q_2,q_3) – find a disk thru points q_1, q_2 and q_3 (easy).



Incremental Algorithm

- MinDisk(P)

- D_2 = the minimal disk through p_1 and p_2.
- For each point p_i in random order ($3 \le i \le n$):
  - If $p_i \in D_{i-1}$ then $D_i = D_{i-1}$ // do nothing
  - Else D_i = MinDisk1(P_{i-1},p_i). // look for other two points on disk
- Return D_n

Incremental Algorithm

- MinDisk1(P,q)

- D_1 = the minimal disk through q and p_1.
- For each point p_i ($2 \le i \le n$):
  - If $p_i \in D_{i-1}$ then $D_i = D_{i-1}$ // do nothing
  - Else D_i = MinDisk2(P_{i-1},q,p_i). // look for other one point on disk
- Return D_n

Incremental Algorithm

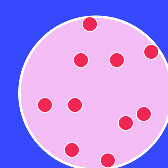
- MinDisk2(P,q_1,q_2)

- D_0 = the minimal disk through q_1 and q_2.
- For each point p_i ($1 \le i \le n$):
  - If $p_i \in D_{i-1}$ then $D_i = D_{i-1}$ // do nothing
  - Else D_i = Disk(q_1,q_2,p_i). // form disk
- Return D_n

Complexity Analysis

- Use backward analysis on point ordering.
- Total time complexity:

$$\sum_{i=1}^{n} O(i)\frac{3}{i} = O(n)$$



- Linear expected runtime.
- Worst case: $O(n^3)$.