CS 445

Dynamic Programming

Some of the slides are courtesy of Charles Leiserson with small changes by Carola Wenk

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Example: Floyd Warshll Algorithm: Computing all pairs shortest paths

- Given $G(V,E)$, with weight $w(v_i, v_j)$ given on each of its edges (positive or negative), the output is a matrix $D[1..n, 1..n]$ such that (for every i,j) $D[i,j]$ is the length of the shortest path from v_i to v_j
- How to find the shortest paths (and not only their costs) will be discussed in in the homeworks. (analogous to Dijkstra)
- Assume no negative cycles exist in $G(V,E)$.
- In the homework: Finding such cycles.

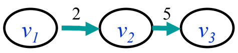
Assume $V=(v_1, v_2 \ldots v_n)$

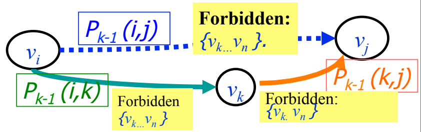
Def $P_k(i,j)$ is the shortest path v_i to v_j avoiding any vertex from $\{v_{k+1 \ldots} v_n\}$ as intermediate vertex.
Example: **$P_k(i,j)$ could not go through any vertex of V.**

Def $D_k[i,j]$ is its length of $P_k(i,j)$

So if the edge (v_i, v_j) is in G then
$P_0(i,j)=\{(v_i, v_j)\}$
$D_0(i,j)=w(v_i, v_j)$

If the edge (v_i, v_j) is not in E, then $D_0(i,j)=+\infty$ (since any path connecting them must use a vertex from $V=\{v_{1 \ldots} v_n\}$





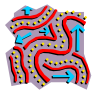
Def $P_k(i,j)$ is the shortest path from v_i to v_j **avoiding** any vertex from $\{v_{k+1...}v_n\}$ as an intermediate vertex. (the sets $\{v_{k+1...}v_n\}$ is forbidden)

Def $D_k[i,j]$ **is its length of** $P_k(i,j)$

- Assume $D_{k-1}[i,j]$ has been computed ($1 < i, j < n$).

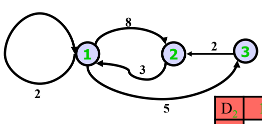
- We now want to compute the matrix $D_k[i,j]$.
- Now we **could** (but don't have to) go through v_k along the shortest path $v_i \rightarrow v_j$.
  - Two option:
  1. Going through v_k is longer, and we better stick to $P_{k-1}(i,j)$. (previous found shortest path $v_i \rightarrow v_j$). Or
  2. Use $P_{k-1}(i,k)$, the shortest path $v_i \rightarrow v_k$ to reach v_k, and continue $P_{k-1}(k,j)$ along to v_j.
- ***Conclusion:*** $D_k[i,j] = \min(D_{k-1}[i,j],\ D_{k-1}[i,k] + D_{k-1}[k,j])$

Floyd Warshll-Pairs Shortest Paths
Computing $D_k[i,j]$ for every i,j,k.



Algorithm *AllPair*(*G*) for all vertex pairs (i,j)
Use n tabels $D_{0....}D_n$. Each is an $n \times n$
if $i = j$ then $D_0[i,i] \leftarrow 0$
else if (v_i, v_j) is an edge in G
  $D_0[i,j] \leftarrow w(v_i, v_j)$
else
  $D_0[i,j] \leftarrow +\infty$
for $k \leftarrow 1$ to n do
 for $i \leftarrow 1$ to n do
  for $j \leftarrow 1$ to n do
   $D_k[i,j] = \min\{\ D_{k-1}[i,j],\ D_{k-1}[i,k] + D_{k-1}[k,j]\ \}$
 return D_n

Floyd's algorithm: example



D_0	1	2	3
1	0	8	5
2	3	0	∞
3	∞	2	0

D_1	1	2	3
1	0 (0+0)	8 (0+8)	5 (0+5)
2	3 (3+0)	0 (3+8)	8 (3+5)
3	∞ (∞+0)	2 (∞+8)	0 (∞+5)

D_2	1	2	3
1	0 (8+3)	8 (8+0)	5 (8+8)
2	3 (0+3)	0 (0+0)	8 (0+8)
3	5 (2+3)	2 (2+0)	0 (2+8)

D_3	1	2	3
1	0 (5+5)	7 (5+2)	5 (5+0)
2	3 (8+5)	0 (8+2)	8 (8+0)
3	5 (0+5)	2 (0+2)	0 (0+0)

Floyd Warshll-Pairs Shortest Paths Computing $D_k[i,j]$ for every i,j,k.

Algorithm *AllPair*(G) for all vertex pairs (i,j)
Use n tabels $D_0 \ldots D_n$. Each is an $n \times n$
if $i = j$ then $D_0[i,i] \leftarrow 0$
else if (v_i, v_j) is an edge in G
  $D_0[i,j] \leftarrow w(v_i, v_j)$
else
  $D_0[i,j] \leftarrow +\infty$
for $k \leftarrow 1$ to n do
  for $i \leftarrow 1$ to n do
    for $j \leftarrow 1$ to n do
      $D_k[i,j] = \min\{ D_{k-1}[i,j], D_{k-1}[i,k] + D_{k-1}[k,j] \}$
return D_n

Running time $O(n^3)$

Space ???

Dynamic Programming: Example 2: Longest Common Subsequance

We look at sequences of characters (strings)

e.g. *x="ABCA"*

Def: A **subsequence** of x is an sequence obtained from x by possibly deleting some of its characters (but without changing their order

Examples:
"ABC", "ACA", "AA", "ABCA"

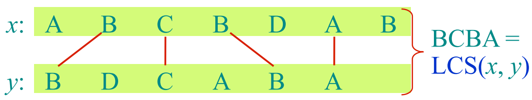
Def A **prefix** of x, denoted $x[1..m]$, is the sequence of the first m characters of x

Examples:
$x[1..4]$="ABCA" $x[1..3]$="ABC" $x[1..2]$="AB"
$x[1..1]$="A" $x[1..0]$=""

Example 1: *Longest Common Subsequence (LCS)*

- Given two sequences $x[1 \,.\,.\, m]$ and $y[1 \,.\,.\, n]$, find a longest subsequence common to them both.

"a" *not* "the"



Different phrasing: Find a set of a maximum number of segments, such that
- Each segment connects a character of x to an identical character of y,
- Each character is used at most once
- Segments do not intersect.

Brute-force LCS algorithm

Checking every subsequence of x whether it is also a subsequence of y.

Analysis

- Checking = $\Theta(m+n)$ time per subsequence.
- 2^m subsequences of x

Worst-case running time = $\Theta((m+n)2^m)$
= exponential time.

Towards a better algorithm

Simplification:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

Notation: Denote the length of a sequence s by $|s|$.

Strategy: Consider ***prefixes*** of x and y.

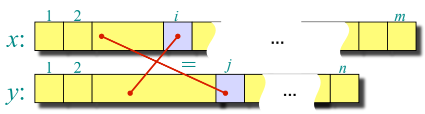
- Define $c[i, j] = |\text{LCS}(x[1 \,..\, i], y[1 \,..\, j])|$.
- Then, $c[m, n] = |\text{LCS}(x, y)|$.

Recursive formulation

Theorem.

$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

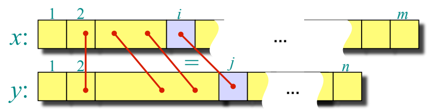
Proof: It is impossible that
x[i] is matched to an element in *y[1..j-1]* and in addition
y[j] is matched to an element in *x[1..i-1]*



Recursive formulation-cont

Case (I): $x[i] = y[j]$. Claim: $c[i, j]=c[i-1,j-1]+1$.

Proof.



We claim that there is a max matching that matches $x[i]$ to $y[j]$.

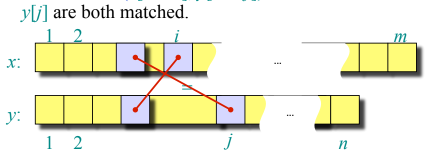
Indeed, if $x[i]$ is matched to $y[k]$ (for $k<j$) then $y[j]$ is unmatched (otherwise we have two crossing segments). Hence we can obtain another matching of the same cardinality by match $x[i]$ to $y[j]$.

This implies that we can match $x[1..i-1]$ to y[$1..j-1$], and add the match ($x[i],y[j]$). So $c[i, j]=c[i-1,j-1]+1$

Recursive formulation-cont

Case (II): $x[i] \neq y[j]$ Claim: $c[i, j]=\max\{c[i-1, j], c[i, j-1]\}$

Recall - in LCS($x[1..i], y[1..j]$) it cannot be that **both** $x[i]$ and $y[j]$ are both matched.



If $x[i]$ is unmatched then

LCS($x[1..\mathbf{i}], y[1..\mathbf{j}]$)= LCS($x[1..\mathbf{i\text{-}1}], y[1..\mathbf{j}]$)

If $y[j]$ is unmatched then

LCS($x[1..\mathbf{i}], y[1..\mathbf{j}]$)= LCS($x[1..\mathbf{i}], y[1..\mathbf{j\text{-}1}]$)

So $c[i, j]= \max\{c[i-1, j], c[i, j-1]\}$

Dynamic-programming hallmark #1

Optimal substructure

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

If z = LCS(x, y), then any prefix of z is an LCS of a prefix of x and a prefix of y.

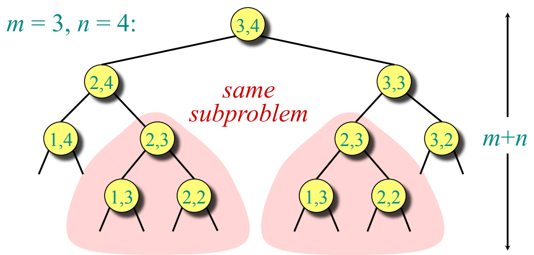
Recursive algorithm for LCS

LCS(x, y, i, j)
    if (i==0 **or** j=0) return 0
    if $x[i] = y[j]$
        then return LCS(x, y, i–1, j–1) + 1
        else return max { LCS(x, y, i–1, j),
                                  LCS(x, y, i, j–1) }

To call the function LCS(x, y, m,n)

Worst-case: $x[i] \neq y[j]$, for all i,j in which case the algorithm evaluates two subproblems, each with only one parameter decremented.

Recursion tree



Height = $m + n$ ⇒ work potentially 2^{m+n} exponential.
but we' re solving subproblems already solved!

Dynamic-programming hallmark #2

Overlapping subproblems
A recursive solution contains a "small" number of distinct subproblems repeated many times.

The number of distinct LCS subproblems for two strings of lengths m and n is only mn.

Memoization algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

```
LCS(x, y)
    for i=0 to m    c[i, 0] = 0
    for j=0 to n    c[0, j] = 0

    for i=1 to m
      for j=1 to n
        if (x[i] = y[j] )
          then c[i, j] ← c[ i–1, j–1]  + 1
          else c[i, j] ← max{ c[ i–1, j],  c[i, j–1] }
```

Time = $\Theta(mn)$ = constant work per table entry.
Space = $\Theta(mn)$.

LCS: Dynamic-programming algorithm

LCS(X,Y)="BCBA"

X=B D C A B A

Y=A B C B D A B

X= \ Y=		1 A	2 B	3 C	4 B	5 D	6 A	7 B
	0	0	0	0	0	0	0	0
1 B	0	0	1	1	1	1	1	1
2 D	0	0	1	1	1	2	2	2
3 C	0	0	1	2	2	2	2	2
4 A	0	1	1	2	2	2	3	3
5 B	0	1	2	2	3	3	3	4
6 A	0	1	2	2	3	3	4	4

Reconstruction *z=LCS(x,y)*

IDEA: Compute the table bottom-up. Fill *z* backward.

Observation: *c[i;j]≥c[i-1;j]* and *c[i;j] ≥c[i;j-1]*
Proof Sketch: We use a longer prefix, so there are more chars to be match.

LCS Reconstruction:

```
Set i=m;  j=n;  k=c[i;j]
While(k>0){
  if (c[i;j]>c[i-1;j] and c[i;j]>c[i;j-1] ) {
        z[k] = x[i] ;
        i--; j-- ; k--  ;
  }else // c[i;j]=c[i-1;j] or c[i;j]=c[i-1;j]
  if  (c[i;j]==c[i;j-1])  j-- ;
  else i--  ;
}
```

LCS(x,y)="BCBA"

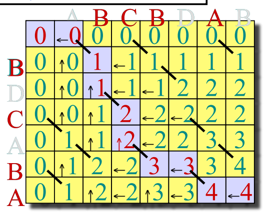
x=B D C A B A

y=A B C B D A B

		1 A	2 B	3 C	4 B	5 D	6 A	7 B
	0	0	0	0	0	0	0	0
1 B	0	0	1	1	1	1	1	1
2 D	0	0	1	1	1	2	2	2
3 C	0	0	1	2	2	2	2	2
4 A	0	1	1	2	2	2	3	3
5 B	0	1	2	2	3	3	3	4
6 A	0	1	2	2	3	3	4	4

Reconstructing *z=LCS(X,Y)*

Another idea – While filling *c[]*, add arrows to each cell *c[i,j]* specifying which neighboring cell *c[i,j]* it got its value.
- *c[i,j].flag* = "\ " if *c[i,j]=c[i-1;j-1]+1*
- *c[i,j].flag* = "↑ " if *c[i,j]=c[i-1;j]*
- *c[i,j].flag* = "←" if *c[i,j]=c[i-1;j]*



Example 3: Edit distance

Given strings *x,y*, the **edit distance** ***ed***(*x,y*) between *x* and *y* is defined as the minimum number of operations that we need to perform on *x*, in order to obtain *y.*

Defintion: An Operations (in this context) Insertion/Deletion/ Replacement of a **single** character.

Examples:
ed("aaba", "aaba") = 0
ed("aaa", "aaba") = 1
ed("aaaa", "abaa") = 1
ed("baaa", "") =4
ed("baaa", "aaab") =2

Example 3' : ``Priced'' Edit distance *ed(x,y)*

Assume also given
InsCost, - the cost of a single **insertion** into *x.*
DelCost - the cost of a single **deletion** from *x,* and
RepCost - the cost of **replacing** one character of *x* by a different character.

Definition: Given strings *x,y*, the **edit distance** ***ed***(*x,y*) between *x* and *y* is the cheapest sequence of operations, starting on *x* and ending at *y.*

Problem: Compute ***ed***(*x,y*), and compute the sequence of operations.

Thm:

Let $c[i,j]$ = ed($x[1..i]$, $y[1..j]$).
Assume $c[i-1,j-1]$, $c[i-1,j-1]$, $c[i-1,j]$ are already computed.

If $x[i]=y[j]$ then $c[i,j] = c[i-1,j-1]$
Else // $x[i] \neq y[j]$

$c[i,j]$ = min{
- **$c[i-1,j-1]$**+*RepCost*, //convert $x[1..i-1]$ ➔ $y[1..j-1]$, and replace $y[j]$ by $x[i]$
- **$c[i-1, j]$** +*DelCost*, //delete $x[i]$ and convert $x[1..i-1]$ ➔ $y[1..j]$
- **$c[i,j-1]$**+*InsCost* //convert $x[1..i]$ ➔ $y[1..j-1]$, and insert $y[i]$

}
}

Algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

```
ed(x, y)
   for i=0 to m   c[i, 0] = i DelCost
   for j=0 to n   c[0, j] = j InsCost

   for i=1 to m
     for j=1 to n
       if (x[i] == y[j] )
         then c[i, j] ← c[ i–1, j–1]
         else c[i, j] ←min{    c[ i-1 , j ]   +     DelCost,
                               c[ i-1, j-1 ] +     RepCost,
                               c[ i , j-1]   +     InsCost
                               }
```

Time = $\Theta(m\ n)$ = constant work per table entry. Space = $\Theta(m\ n)$.