CSc445 Algorithms

Quick Sort and median selection

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Based on slides curacy of

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QuickSort – example of the divide-and-concourse paradigm

- Proposed by C.A.R. Hoare in 1962.
- Sorts "in place" (no need for extra space). Like insertion sort, but not like merge sort.
- Very practical (with tuning).

Divide and conquer

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\le x \le$ elements in upper subarray.

$\le x$	x	$\ge x$

2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

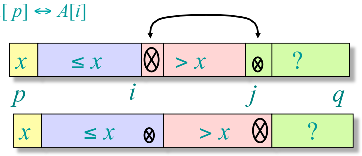
Key: *Linear-time partitioning subroutine.*

Partitioning subroutine

```
PARTITION(A, p, q)  ⊳ A[ p . . q]
   x ← A[ p]        ⊳ pivot = A[ p]
   i ← p
   for j ← p + 1 to q
      do if A[ j] ≤ x          ⊳ Should send A[ j] to the left.
           then{
                 i ← i + 1              ⊳ Now A[i]>x
                 exchange A[i] ↔ A[ j] ⊳ Fix A[i]>x
           }
   exchange A[ p] ↔ A[i]
   return i
```

Running time = $O(n)$ for n elements.

Invariant:



Example of partitioning

6	10	13	5	8	3	2	11
i	j						

L4.5

Example of partitioning

6	10	13	5	8	3	2	11
i		$\to j$					

L4.6

Example of partitioning

6	10	13	5	8	3	2	11
i			*j*				

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
	i		*j*				

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
	i			*j*			

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
	i				$\longrightarrow j$		

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
		$\longrightarrow i$			j		

L4.11

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
		i				$\longrightarrow j$	

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11
			i			j	

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
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			i				j

Example of partitioning

6	10	13	5	8	3	2	11	
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6	5	3	2	8	13	10	11	
			i					j

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11
2	5	3	6	8	13	10	11
			i				

Pseudocode for quicksort

QUICKSORT(A, p, r)
    if $p < r$
        then $q \leftarrow$ PARTITION(A, p, r)
            QUICKSORT($A, p, q-1$)
            QUICKSORT($A, q+1, r$)

Initial call: QUICKSORT(A, 1, n)

Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.

Worst-case of quicksort

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \qquad \textbf{\textit{(arithmetic series)}} \end{aligned}$$

Worst-case recursion tree

$T(n) = T(0) + T(n–1) + cn$

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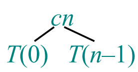
$T(n)$

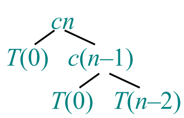
Worst-case recursion tree

$T(n) = T(0) + T(n-1) + cn$



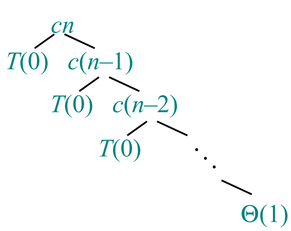
Worst-case recursion tree

$T(n) = T(0) + T(n-1) + cn$



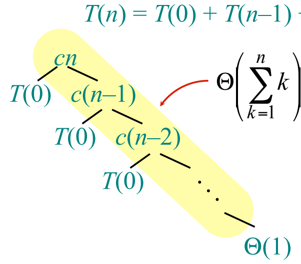
Worst-case recursion tree

$T(n) = T(0) + T(n-1) + cn$



Worst-case recursion tree

$T(n) = T(0) + T(n-1) + cn$

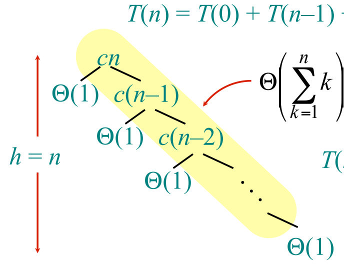


$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta(n^2)$$

L4.25

Worst-case recursion tree

$T(n) = T(0) + T(n-1) + cn$



$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta(n^2)$$

$$\begin{aligned} T(n) &= \Theta(n) + \Theta(n^2) \\ &= \Theta(n^2) \end{aligned}$$

Best-case and almost best-case analysis

If we are lucky, PARTITION splits the array evenly:

$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned} \qquad \text{(same as merge sort)}$$

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?

L4.27

Analysis of “almost-best” case

$T(n)$

L4.28

Analysis of “almost-best” case

$$cn$$

$$T\left(\tfrac{1}{10}n\right) \qquad T\left(\tfrac{9}{10}n\right)$$

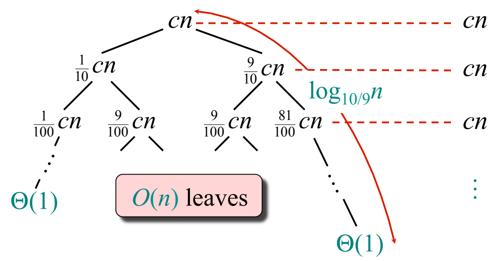
Analysis of “almost-best” case

$$cn$$

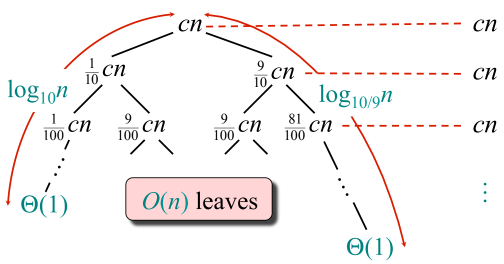
$$\tfrac{1}{10}cn \qquad \tfrac{9}{10}cn$$

$$T\left(\tfrac{1}{100}n\right) T\left(\tfrac{9}{100}n\right) \quad T\left(\tfrac{9}{100}n\right) T\left(\tfrac{81}{100}n\right)$$

Analysis of "almost-best" case



Analysis of "almost-best" case



$$cn\log_{10}n \le T(n) \le cn\log_{10/9}n + O(n)$$
$$\le 8\ c\ \log_2 n$$

Randomized quicksort

How can find a pivot that guarantees partitions with good ratios for *A[1..n]*, ?

We say that q is a **good pivot** for if

- at least 10% of the elements of *A[1..n]* are smaller than q, and
- at least 10% of the elements of *A[1..n]* are larger than q.

$10\% \le q$		$10\% \ge q$

Best pivot: Pick the median of *A[1..n]*, as pivot.

(median – an element that is larger than half of the elements)

Then the time would obey $T(n) = cn+2T(n/2)$

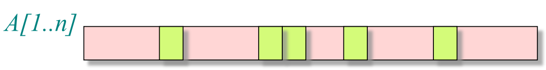
Problem – need to work too hard to find the median (best pivot), so we will do with (only) a good pivot.

Finding a good pivot for *A[1..n]*

5-random-elements method. :

- Pick the **indices** of 5 elements at random from *A[1..n]*,
- *For k=1 to 5*

  X[k] = A[n rnd()]



- Set q to be the median of *X[1..5]*

Finding a good pivot for *A[1..n]*

5-random-elements method. : Pick 5 elements at random from *A[1..n]*, and set q to be their median.

What it is the probability that q is **not** a good pivot ?

- Let S be the elements of *A[1..n]* which are the 10% smallest.
- The probability that an elements picked at random is in S is *0.1*.
- q is in S only if at least 3 of the 5 elements that we pick are in S.
- The probability that this happens is

$0.1^5 +$ (all in S) $5 \bullet 0.1^4 \bullet 0.9 +$ (4 in S, one not in S) $10 \bullet 0.1^3 \bullet 0.9^2 =$ (2 not in S)

$= 0.00001 + 0.00045 + 0.00810 = 0.00856$

- This is also the probability that q is in the 10% largest elements.
- In other words: with probability ≥98%, q is a good pivot.

Randomized quicksort – cont
Finding good pivots

Putting it together, during QS, each time that we need to find a pivot, we use the "5 random elements" method.

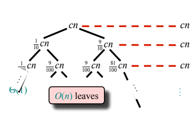
With probability 98%, we find a good pivot.

The overall time that we spend on good partitions is much smaller than the time we spent on bad partitions.

(note – bad partitions are not harmful – they are just not helpful)

So the recursions formula $T(n) = cn + T(n/10) + T(n\,9/10)$ still apply, leading to running time $O(n \log n)$.

This is expected running time – there is a chance that the actual running time is $\Theta(n^2)$, but the probability that it happens is very slim.



Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort behaves well even with caching and virtual memory.

Median Selection

- (CLRS Section 9.2, page 185).
- For $A[1..n]$ (all different elements) we say that the rank of x is $\boldsymbol{i}$ if exactly $\boldsymbol{i-1}$ elements in A are smaller than x.
- In particular, the median is the $\lfloor n/2 \rfloor$-smallest.
- To find the median, we could sort and pick $A[\lfloor n/2 \rfloor]$ (taken $O(n \log n)$).
- We can do better.

L4.38

Median Selection-cont

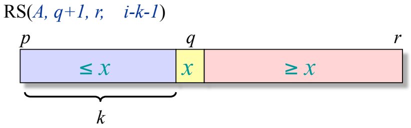
RS(A, p, r, i){

//**R**andomize **S**election: Returns i'st smallest element in $A[p..r]$.

//Assumption: Input is valid and elements are different.

- If $p==r$ return A[p]
- q=PARTITION(A,p,r) ;
  - //Partition using the 5-random element method
- $k=q-p$
- If $i==k+1$ return $A[q]$
- If $i<k$ return RS($A, p, q-1, i$) // Note the difference from QS
- Else return RS($A, q+1, r, i-k-1$)

}



Time analyis

- Recall: With high probability, we pick a good pivot:
  - Not in the 10% smallest or largest:
- Hence, we get rid of at least 10% of the elements of *A*
- So, $T(n)=cn+T(0.9\,n)$.
  - $T(n)=c(n+0.9n+0.9^2n+0.9^3n+\ldots) =$
    $cn(1+0.9+0.9^2+0.9^3+\ldots) =$
    $cn(1/(1-0.9)) = O(n)$.
- So the expected time is linear. (yuppie)

As in the case of QS, partitions which are not good are not harmful, just not helpful.