SkipList

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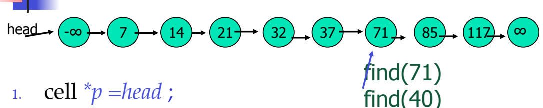
Searching a key *x* in a sorted linked list



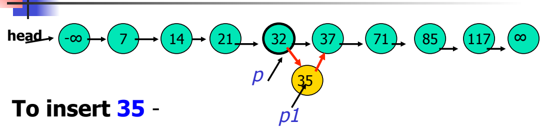
1. cell **p =head ;*
2. while *(p->key < x) p=p->next ;*
3. return p ; / /(which is either equal or larger than ***x***)

Note:

- The -∞ and ∞ elements are not "real" keys.
  - They are in the list to prevent checking special cases
- Sometimes we prefer to return the element proceeding the one containing ***x***. **Then line 2 is replaced with**

  while *(p->>next->key < x) p=p->next*

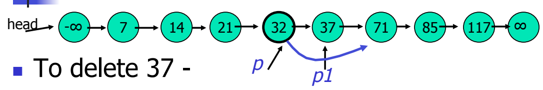
inserting a key into a Sorted linked list



To insert 35 -

- *p*= find*(35); // find the proceeding element – the next one is > 35*
- CELL **p1* = (CELL *) malloc(sizeof(CELL));
- *p1->key=35;*
- *p1->next = p->next ;*
- *p->next = p1 ;*

deleteing a key from a sorted list

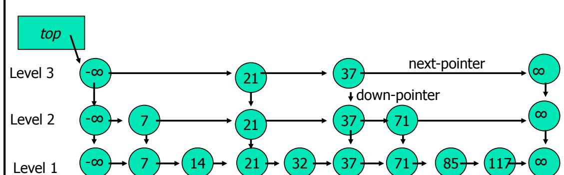


- To delete 37 -
- *p=find(37); // Again find proceeding element*
- CELL **p1 =p->next;*
- *p->next = p1->next ;*
- free(*p1*);

SKIP LIST - A data structure for maintaining keys in a sorted order

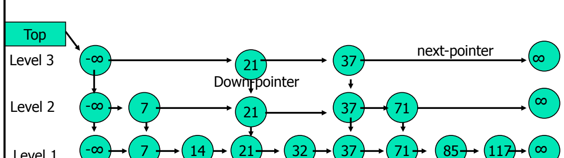
Rules:

- Consists of several **levels.**
- All keys appear in level 1
- Each level is a sorted list.
- If key ***x*** appears in level ***i***, then it also appears in all levels below level ***i***
- First element in each level has key -∞ .
- Last element has key +∞
- First element in upper level is pointed to by variable ***top.***

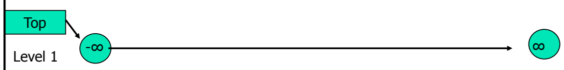


More rules

- An element in level ***i >1*** points (via down pointer) to the element with the same key in the level below.
- Elements in the lowest level have ***down-pointer=NULL***
- Also maintain a counter specifying the number of levels.

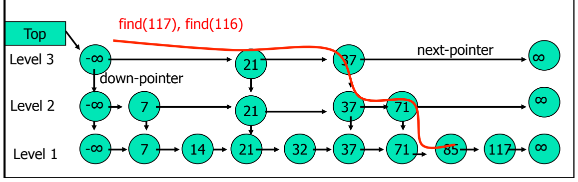


An empty SkipList



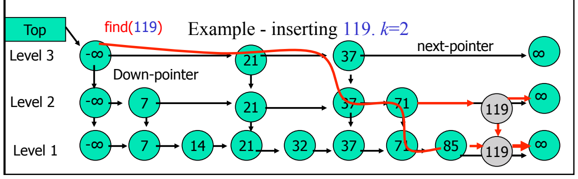
Finding an element with key x

- *p=top* ;
- while(1){
  - while (*p*➔*next*➔*key* < ***x***) *p=p*➔*next*;
  - if (*p*➔*down* == *NULL*) **return** *p*
    - // Note: This returns the element itself. If interested in the predecessor, return *p.*
  - *p=p*➔*down* ;
- }
- Observe that we return *pred(**x**)* (the key proceeding *x*).



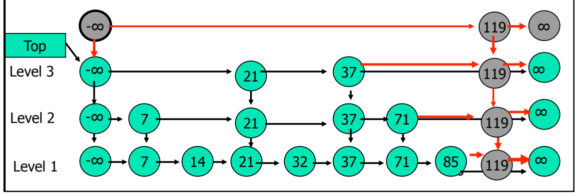
Inserting new element x

- Determine ***k*,** defined as the number of levels in which ***x*** participates (explained later how)
- Perform find(***x***), but once the search path is in one of the lowest ***k*** levels:
  - ***x*** is inserted after the elements at which the search path branches down or terminates.
  - The *next-pointer* behave like a "standrad" linked list
  - The *down pointer(s)* point between themselves.



Inserting an element - cont.

- If ***k*** is larger than the current number of levels, add new levels (and update *top*, and *num_of_levels* counter)
- Example - **insert(119)** when ***k=4***
- `Heuristic: Add at most one new level (not needed for the analysis)`

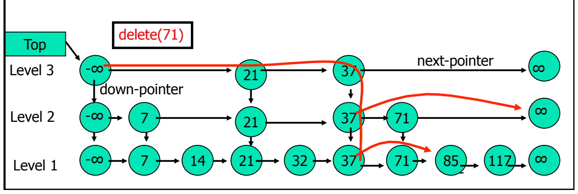


Determining *k*

- ***k*** **-** the number of levels at which an element *x* participate.
- Use a random function *OurRnd()* --- returns 1 or 0 (True/False) with equal probability.
  - ***k****=1 ;*
  - *While(OurRnd()==1)* ***k****++ ;*

Deleteing a key *x*

- Find ***x*** in all the levels it participates, using find(***x***).
- During the "find", delete ***x*** from each level it participates using the standard "delete from a linked list" method.
- If one or more of the upper levels become empty, remove them (and update *top* and *num_of_levels*)



"expected" on what ?

- **Claim:** The expected number of elements is O(n).
- he term "**expected**" here refers to the experiments we do while tossing the coin (or calling *OurRnd()*). No assumption about input distribution.
- So imagine a given set, given set of operations insert/del/find, but we repeat many time the experiments of
  - constructing the SL, and count the #elements.

Facts about SL

- **Def:** The **height** of the SL is the number of levels
- **Claim:** The expected number of levels is O(log n)
- (here n is the number of keys)
- **"≅ Proof"**
  - The number of elements participate in the lowest level is n.
  - Since the probability of an element to participates in level 2 is ½, the **expected** number of elements in level 2 is **n/2**.
  - Since the probability of an element to participates in level 3 is $1/4$, the expected number of elements in level 3 is $n/4$.
  - …
  - The probability of an element to participate in level j is $(1/2)^{j-1}$ so number of elements in this level is $n/2^{j-1}$
  - So after $\log(n)$ levels, no element is left.

Facts about SL

- **Claim:** The expected number of elements is O(n).
- (here n is the number of keys)
- **"≅ Proof"** (a rigorous proof requires the use of random variables)
  - The total number of elements is

    $n+n/2+n/4+n/8\ldots \le n(1+1/2+1/4+1/8\ldots) = 2n$

    To reduce the worst case scenario, we verify during insertion that k (the number of levels that an element participates) in) is $\le \log n$.

Conclusion: The expected storage is $O(n)$

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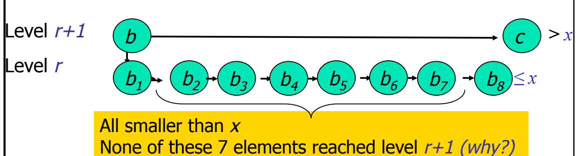
More facts

- **Thm**: The expected time for find/insert/delete is O(log *n*)
- **Proof** For all Insert and Delete, the time is ≤ expected #elements scanned during find(x) operation.
- Will show: Need to scan expected O(log *n*) elements.

Thm: Expected time for `**find'** operation is O(log *n*)

- **Proof** – we know that there are *O(* log *n)* levels. Will show that we spend O(1) time in each level.
- Assume during find(x), we scanned t elements, (for $t>8$) in level r. Assume first that r is not the upper level.
  - (the search visited b, branched down to b_1 and then visited $b_2 \ldots b_8$ (not sure what happed before or after)



The probability that none of these 7 elements reached level $r+1$ is $1/2^7$. For larger value of 7 – very slim.

Bounding time for insert/delete/find

- Putting it together The expected number of elements scanned in each level is O(1)
- There are O(log n) levels
- Total time is O(log *n*)
- As stated, getting bounds for time for insert/delete are similar

How likely is that the SL is too tall ?

- Lets ask how likely it is that the #levels is
  $Z\log_2 n$, where $Z=1,2,3...$

That is, we estimate the probability that the height of the SL is

- $\log_2 n$
- $2 \log_2 n$
- $3 \log_2 n$
- $4 \log_2 n$
  - ...

Reminder from probability

- Assume that ***A,B*** are two events. Let
  - Pr(***A***) be the probability that ***A*** happens,
  - Pr(***B***) be the probability that ***B*** happens
  - Pr(***A***$\cup$ ***B***) is the probability that either event ***A*** happens or event ***B*** happens (or both).
- So probably that at least one of them happened is
  $\Pr(A)+\Pr(B)-\Pr(A \cap B) \leq \Pr(A)+\Pr(B)$

Similarly, for 3 Events A_1, A_2, A_3. The probability that **at least** one of them happens

$\Pr(A_1 \cup A_2 \cup A_3) \leq \Pr(A_1)+\Pr(A_2)+\Pr(A_3)$

Example: In a roulette, we pick a number ***k*** between 1..38

- Event **A**: ***k*** is even. Pr(***A***)=Pr(***k*** is even) = 19/38 = 0.5
- Event **B** : ***k*** is divided by 3. Pr(***B***)= 12/38=0.315
- Pr(**A** or **B**) = Pr((***k*** is divided by 2) or (***k*** is divided by 3))
  $\leq$0.5+0.3=0.8

But how likely is that the SL is too tall ?

- Assume the keys in the SL are $\{x_1, x_2, \dots x_n\}$
- The probability that x_1 participates in at least k levels is 2^{-k}
- (same probability for all x_i).
- Define: A_1 is the event that x_1 *participates in* $\geq k$ *levels.*
- $\Pr(A_1) \leq 2^{-k}$
- Define: A_j is the event that x_j *participates in* $\geq k$ *levels*
- $\Pr(A_j) \leq 2^{-k}$
- If the height of SL $\geq k$ *then*
  at least one of the x_j participate in $\geq k$ levels.
- The probability that **any** x_i participates in $\geq k$ levels is $\leq$
  $\Pr(A_1) + \Pr(A_2) + \dots + \Pr(A_n) = n\, 2^{-k}$
- **This is the probability that the height of the *SL is* $\geq k$**

But how likely is that the SL is tall ?

- The probability that **any** x_i participates in at least k levels is $\leq n2^{-k}$. Then the height of the SL $\geq k$.
- Recall $y^{(ab)}=(y^a)^b$.
- Write $k= Z \log_2 n$, and recall that $2^{\log n} = n$.
- Want to find: The probability that the height is Z times $\log_2 n$.
  - Twice, 3 time, 4 times…
- Then $2^{-k} = 2^{-(Z \log n)} = (2^{\log n})^{-Z} = n^{-Z} = 1/n^Z$
- So $n2^{-k} \leq n / n^Z = 1/n^{Z-1}$
- This is the probability that the height of SL $\geq Z \log_2 n$
- Example: $n=1000$.
- The probability that the heigh$\geq 7 \log_2 n$ is $\leq 1/ 1000^6$ $1/10^{18}$
- The prob. that the heigh$\geq 10 \log_2 n$ is $\leq 1/ 1000^9 = 1/10^{27}$

In other words (and with some hand-waving)

- Assume we have a set of $n>1000$ keys, and we keep rebuilding Skiplists for them.
- Call a SL ***bad*** if its height > $7 \log_2 n$
- First build SL_1
- Then build SL_2 (for the same keys)
- Then …
- Then SL_M where $M=10^{20}$
- Then less than 100 of them are bad.

- Using Similar techniques we can also bound the probability that the search takes more than $Z \log_2 n$