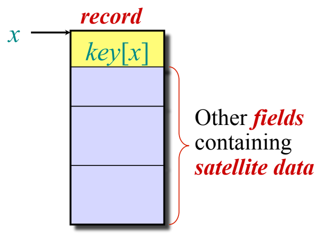
Hashing

Thanks to
Prof. Charles E. Leiserson

Symbol-table problem

Symbol table T holding n ***records***:



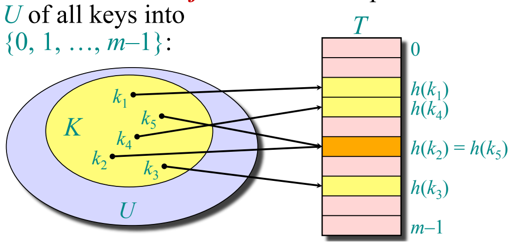
Operations on $\boldsymbol{T}$:

- INSERT(T, x)
- DELETE(T, x)
- SEARCH(T, k)

How should the data structure T be organized?

Hash functions

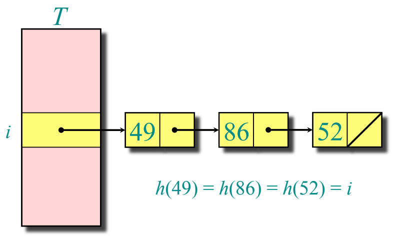
Idea: Use a ***hash function*** h to map the universe U of all keys into $\{0, 1, \ldots, m-1\}$:



When a record to be inserted maps to an already occupied slot in T, a ***collision*** occurs.

Resolving collisions by chaining

- Records in the same slot are linked into a list.



Analysis of chaining

We make the assumption of ***simple uniform hashing:***

- Each key $k \in K$ of keys is equally likely to be hashed to any slot of table T, independent of where other keys are hashed.

Let n be the number of keys in the table, and let m be the number of slots.

Define the ***load factor*** of T to be

$$\alpha = n/m$$

$= \text{average number of keys per slot.}$

Search cost

Expected time to search for a record with a given key $= \Theta(1 + \alpha)$.

apply hash function and access slot

search the list

Expected search time $= \Theta(1)$ if $\alpha = O(1)$, or equivalently, if $n = O(m)$.

Choosing a hash function

The assumption of simple uniform hashing is hard to guarantee, but several common techniques tend to work well in practice as long as their deficiencies can be avoided.

Desirata:

- A good hash function should distribute the keys uniformly into the slots of the table.
- Regularity in the key distribution should not affect this uniformity.
- Hope: if $k_1 \neq k_2$ in **any** bit, then $h(k_1) \neq h(k_2)$

Division method

Assume all keys are integers, and define
$h(k) = k \bmod m.$

Deficiency: Don't pick an m that has a small divisor d. A preponderance of keys that are congruent modulo d can adversely affect uniformity.

Extreme deficiency: If $m = 2^r$, then the hash doesn't even depend on all the bits of k:

- If $k = 1011000111\underbrace{011010}_{h(k)}{}_2$ and $r = 6$, then $h(k) = 011010_2$.

Division method (continued)

$$h(k) = k \bmod m.$$

Pick m to be a prime not too close to a power of 2 or 10 and not otherwise used prominently in the computing environment.

Annoyance:

- Sometimes, making the table size a prime is inconvenient.

But, this method is popular, although the next method we'll see is usually superior.

Multiplication method

Assume that all keys are integers, $m = 2^r$, and our computer has w-bit words. Define

$$h(k) = (A \cdot k \bmod 2^w) \text{ rsh } (w - r),$$

where rsh is the "bit-wise right-shift" operator and A is an odd integer in the range $2^{w-1} < A < 2^w$.

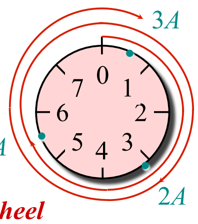
- Don't pick A too close to 2^w.
- Multiplication modulo 2^w is fast.
- The rsh operator is fast.

Multiplication method example

$$h(k) = (A \cdot k \bmod 2^w) \text{ rsh } (w - r)$$

Suppose that $m = 8 = 2^3$ and that our computer has $w = 7$-bit words:

$$\begin{array}{rl} 1011001 & = A \\ \times \quad 1101011 & = k \\ \hline 1001010\underbrace{011}_{h(k)}0011 & \end{array}$$



Modular wheel

Dot-product method

Randomized strategy:

Let m be prime. Decompose key k into $r + 1$ digits, each with value in the set $\{0, 1, \ldots, m-1\}$. That is, let $k = \langle k_0, k_1, \ldots, k_r \rangle$, where $0 \le k_i < m$.

Pick $a = \langle a_0, a_1, \ldots, a_k \rangle$ where each a_i is chosen randomly from $\{0, 1, \ldots, m-1\}$.

Define $h_a(k) = \sum_{i=0}^{r} a_i k_i \bmod m$.

- Excellent in practice, but expensive to compute.

Resolving collisions by open addressing

No storage is used outside of the hash table itself..

- The hash function depends on both the **key** and **probe number**:
  $h : U \times \{0, 1, \ldots, m-1\} \rightarrow \{0, 1, \ldots, m-1\}$.
  E.g. $h(k,i) = (k+i) \bmod m$; $h(k,i) = (k+i^2) \bmod m$

  Inserting a key k:
  we check $T[h(k,0)]$. If empty we insert k, there. Otherwise,
  we check $T[h(k,1)]$. If empty we insert k, there. Otherwise,…
  otherwise etc for $h(k,2), h(k,3), \ldots, h(k,m-1)$.

  Finding a key k:
  we check if $T[h(k,0)] = k$. If not, if empty, stop. otherwise
  we check if $T[h(k,1)] = k$. If not, if empty, stop. otherwise
  otherwise etc for $h(k,2), h(k,3), \ldots, h(k,m-1)$.

  Deleting a key k
  Find it are replace with a dummy – NIL (why ?)

Maintenance

Scan the table from time to time, and get rid of all of all dummies.

Resolving collisions by open addressing - cont

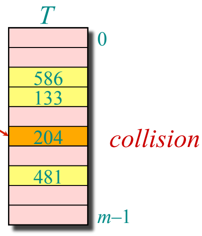
No storage is used outside of the hash table itself.

- The probe sequence $\langle h(k,0), h(k,1), \ldots, h(k,m-1)\rangle$ should be a permutation of $\{0, 1, \ldots, m-1\}$.
- The table may fill up, and deletion is difficult (but not impossible).

Example of open addressing

Insert key $k = 496$:

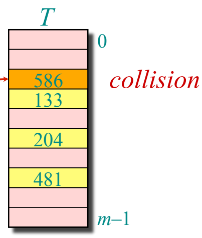
0. Probe $h(496,0)$



Example of open addressing

Insert key $k = 496$:

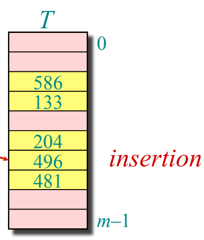
0. Probe $h(496,0)$
1. Probe $h(496,1)$



Example of open addressing

Insert key $k = 496$:

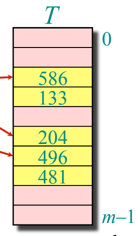
0. Probe $h(496,0)$
1. Probe $h(496,1)$
2. Probe $h(496,2)$



Example of open addressing

Search for key $k = 496$:

0. Probe $h(496,0)$
1. Probe $h(496,1)$
2. Probe $h(496,2)$



Search uses the same probe sequence, terminating successfully if it finds the key and unsuccessfully if it encounters an empty slot.

Slot contains a NIL is not considered empty.

Probing strategies

Linear probing:

Given an ordinary hash function $h'(k)$, linear probing uses the hash function

$$h(k,i) = (h'(k) + i) \bmod m.$$

This method, though simple, suffers from ***primary clustering***, where long runs of occupied slots build up, increasing the average search time. Moreover, the long runs of occupied slots tend to get longer.

Probing strategies

Double hashing

Given two ordinary hash functions $h_1(k)$ and $h_2(k)$, double hashing uses the hash function

$$h(k,i) = (h_1(k) + i \cdot h_2(k)) \bmod m.$$

This method generally produces excellent results, but $h_2(k)$ must be relatively prime to m. One way is to make m a power of 2 and design $h_2(k)$ to produce only odd numbers.

Analysis of open addressing

We make the assumption of ***uniform hashing:***

- Each key is equally likely to have any one of the $m!$ permutations as its probe sequence.

Theorem. Given an open-addressed hash table with load factor $\alpha = n/m < 1$, the expected number of probes in an unsuccessful search is at most $1/(1-\alpha)$.

Proof of the theorem

Proof.

- At least one probe is always necessary.
- With probability n/m, the first probe hits an occupied slot, and a second probe is necessary.
- With probability $(n-1)/(m-1)$, the second probe hits an occupied slot, and a third probe is necessary.
- With probability $(n-2)/(m-2)$, the third probe hits an occupied slot, etc.

Observe that $\dfrac{n-i}{m-i} < \dfrac{n}{m} = \alpha$ for $i = 1, 2, \ldots, n$.

Proof (continued)

Therefore, the expected number of probes is

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(1+\frac{n-2}{m-2}\left(\cdots\left(1+\frac{1}{m-n+1}\right)\cdots\right)\right)\right)$$

$$\le 1+\alpha(1+\alpha(1+\alpha(\cdots(1+\alpha)\cdots)))$$

$$\le 1+\alpha+\alpha^2+\alpha^3+\cdots$$

$$= \sum_{i=0}^{\infty}\alpha^i$$

$$= \frac{1}{1-\alpha}. \ \square$$

The textbook has a more rigorous proof.

Implications of the theorem

- If α is constant, then accessing an open-addressed hash table takes constant time.
- If the table is half full, then the expected number of probes is $1/(1–0.5) = 2$.
- If the table is 90% full, then the expected number of probes is $1/(1–0.9) = 10$.

A weakness of hashing

Problem: For any hash function h, a set of keys exists that can cause the average access time of a hash table to skyrocket.

- An adversary can pick all keys from $\{k \in U : h(k) = i\}$ for some slot i.

IDEA: Choose the hash function at random, independently of the keys.

- Even if an adversary can see your code, he or she cannot find a bad set of keys, since he or she does not know exactly which hash function will be chosen.