Credits:
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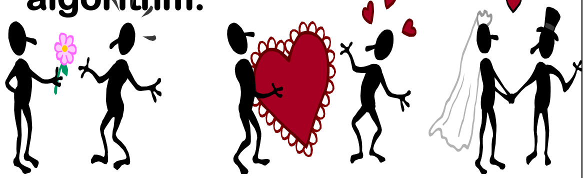
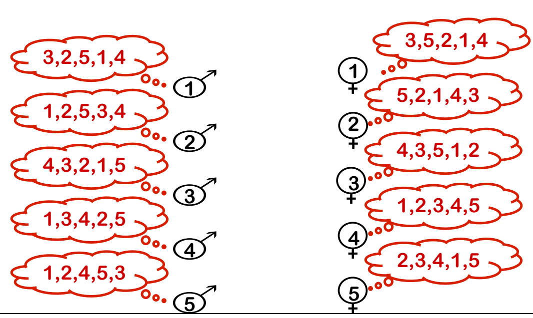
Dating: Who wins the battle of the sexes? Stable marriage (matching) algorithm.



- There are *n* males and *n* females
- Each female has her own ranked preference list of all the males
  - E.g., women #1 most prefers male #3 over any other male.
- Each male has his own ranked preference list of the females
- How should we match them (1-to-1)



Rogue Couples

- Consider a given matching M . Now suppose that some pair (male, female) who are not married to each other, actually prefer each other over their partners.

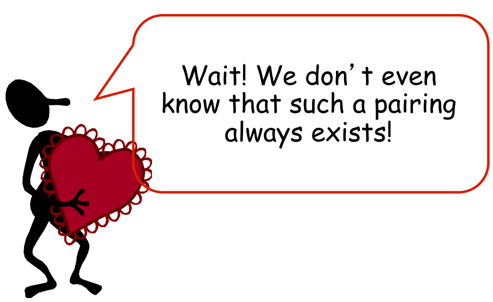
- They will be called a <u>rogue couple</u>.
- They both would gain from dumping their mates and marrying each other.

- A matching is called <u>stable</u> if it does not contains no rogue couples.

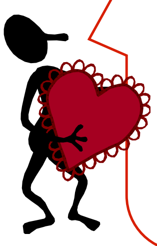
The study of stability will be the subject of the entire lecture.

We will: Analyze various mathematical properties of an algorithm that looks a lot like 1950's dating.

Given a set of preference lists, how do we find a stable pairing?

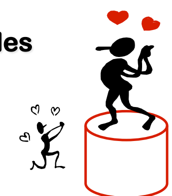


Is there always a stable matching ?



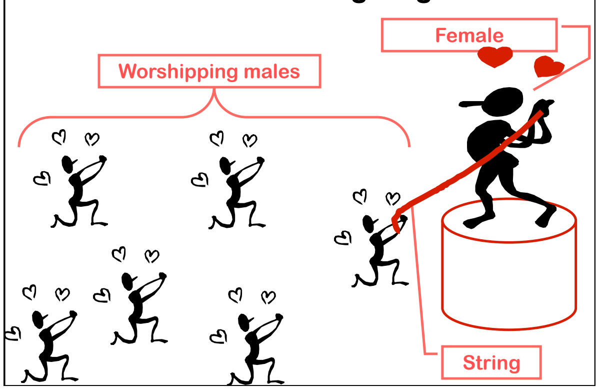
- Will show: every set of preference lists have a stable pairing.
- Will prove it by presenting a fast algorithm that, given any set of input lists, will output a stable pairing.

Terminology and principles



- A male can **propose** (marriage) to a female.
- A female can **reject** the proposal.
- During most of the process, a female would not accept a proposal, but would tell a proposing male "**maybe**".
- This is called "**putting the male on a string**"
- Once a male is rejected, he **crosses** off from his list the rejecting female – he will not propose to her again.
- Once a male proposes, he cannot change his mind until he is rejected.

The Traditional Marriage Algorithm



Traditional Marriage Algorithm (TMA)

1) repeat{

- Morning
  - Each male to the best female whom he has not yet crossed off
- Afternoon (for each females with at least one proposal)
  - To today's best offer: **"Maybe, come back tomorrow" (putting him on a string)**
  - All other proposals are rejected.
- Evening
  - Any rejected male crosses the rejecting female off his list.

}Until all males are on strings.

2) Each female marries the last male she just said "maybe"

Note: Each male proposes to females in decreasing order on his list.

Lemma: If a female has a male *b* on a string, then she will either marry him, or marry someone she prefers over him.

Proof:

- She would only let go of *b* in order to "maybe" *b'* which she prefers over *b*
- She would only let go of *b'* for someone *b''* she prefers over *b'* etc.

When the process terminates, she is left with someone she prefers over *b.*

Corollary: Each female will marry her absolute favorite of the males who visit her during the Traditional Marriage Algorithm (TMA)

Lemma: No male can be rejected by all the females

•Proof by contradiction.

•Suppose male *b* is rejected by all the females. At that point:

- Each female must have a suitor other than ***b*** (By previous Lemma, once a female has a suitor she will always have at least one)
- The ***n*** females have ***n*** suitors, ***b*** not among them. Thus, there are at least ***n+1*** males.

Contradiction

Theorem: The TMA always terminates after at most n^2 days

Proof

- The total length of the lists of all males is $n \times n = n^2$.
- Each day at least one male gets a "No", so at least one female is deleted from one of the lists.
- Therefore, the number of days is bounded by the original size of the master list $= n^2$.

Great! We know that TMA will terminate and produce a pairing.

But is it stable?

<u>Theorem</u>: TMA. Produces a stable pairing.

1. Let m_1 and f_1 be any couple in *T*.
2. Suppose m_1 prefers f_2 over f_1 .
3. We will argue that f_2 prefers her husband over m_1 .
4. During TMA, male m_1 proposed to f_2 before he proposed to f_1 .
5. Hence, at some point f_2 rejected m_1 for someone she preferred.
6. By the Improvement lemma, the man she married was also preferable to m_1
7. Thus, every male will be rejected by any female he prefers to his wife.

8. *T* is stable. QED.

Forget TMA for a moment

- How should we define what we mean when we say "the optimal female for male *b*"?

Flawed Attempt:
"The female at the top of b's list"

The Optimal female

- A male's *optimal female* is the highest ranked female for whom there is <u>some</u> stable matching in which they are married.
- (note – this is **not always** the highest female on his list).

- She is the best female he can conceivably get in a stable world. Presumably, she might be better than the female he gets in the stable pairing output by TMA.

Thm

- The Traditional Marriage Algorithm yields a matching at which each male gets his optimal female

Thm: TMA in a sequential way

- Assume: At each time stamp, (every `tick' of the clock) there is exactly one **event:**
  - Event: a single man proposes, and if got rejected, his next proposal will be in next time stamp)
- Note: The exact order is not crucial:
  - If both m_1, m_2 are proposing to f, the result is the same independent of whom proposed first.

Thm: TMA produces a male-optimal pairing

- **Proof**: Suppose, for a contradiction, that some male gets rejected by his optimal female during TMA.
- Let t be the earliest time at which a male m_1 got rejected by his optimal female f *(Florence)*
- Florence rejected m_1 because she said "maybe" a preferred male m_2.
- m_2 had not yet been rejected by his optimal female (by the definition of t).
- Therefore f is either the optimal female of m_2 Or f is higher the optimal female in his list.

That is, in any stable world, m_2 would either be married to f, or to somebody lower on his list *(definition of opt)*

- Let S be the matching at which (m_1, f) are married
  (S is NOT the result of the TMA)
- Now consider (m_2, f) – they are a **rouge** couple. **QED**

The Pessimal male

- A female's pessimal male is the lowest ranked male for whom there is some stable matching which the female gets him.
- He is the worst male she can conceivably get in a stable world.

Thm: The TMA is female-pessimal.

Proof: We know it is male-optimal. (m_1,f_1) is a couple in *TMA*, => f_1 *is* m_1 optimal female.

Suppose there is a stable pairing S where some female f_1 does worse than in *TMA*.

- Let m_1 be f_1 husband in *TMA*.
- Let m_2 be f_1 husband in S
  (m_2, f_1) is a couple in S (m_2 is worse than m_1)
- By assumption, m_1 prefers f_1 over his wife f_2 in S
  - (since f_1 is his optimal female)
- So (m_1, f_1) is a rogue couple.
- **Therefore, S is not stable. QED**

REFERENCES

- D. Gale and L. S. Shapley, *College admissions and the stability of marriage*, American Mathematical Monthly 69 (1962), 9-15
- Dan Gusfield and Robert W. Irving, *The Stable Marriage Problem: Structures and Algorithms*, MIT Press, 1989