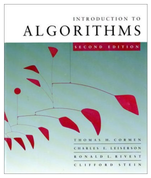
String Matching



Thanks to

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String Matching

- Input: Two strings T[1…n] and P[1…m], containing symbols from alphabet Σ
- Goal: find all "shifts" $0 \leq s \leq n\text{-}m$ such that
  T[s+1…s+m]=P
- Example:
  - Σ={ ,a,b,…,z}
  - T[1…18]="to be or not to be"
  - P[1..2]="be"
  - Shifts: 3, 16

Simple Algorithm

for $s \leftarrow 0$ **to** $n\text{-}m$
    Match $\leftarrow 1$
    for $j \leftarrow 1$ **to** m
        if $T[s+j] \neq P[j]$ **then**
            Match $\leftarrow 0$
            exit loop
    if *Match*=1 **then output** s

Results

- Running time of the simple algorithm:
  - Worst-case: $O(nm)$
  - Average-case (random text): $O(n)$
- Is it possible to achieve $O(n)$ for any input ?
  - Knuth-Morris-Pratt' 77: deterministic
  - Karp-Rabin' 81: randomized

Karp-Rabin Algorithm

- A very elegant use of an idea that we have encountered before, namely…

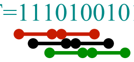
  HASHING !
- Idea:
  - Hash all substrings $T[1\ldots m]$, $T[2\ldots m+1]$, $T[3\ldots m+2]$, etc.
  - Hash the pattern $P[1\ldots m]$
  - Report the substrings that hash to the same value as P
- Problem: how to hash $n-m$ substrings, each of length m, in $O(n)$ time ?

Implementation

- Attempt I:
  - Assume $\Sigma=\{0,1\}$
  - Think about each $T^s=T[s+1\ldots s+m]$ as a number in binary representation, i.e.,

    $$t_s=T[s+1]2^0+T[s+2]2^1+\ldots+T[s+m]2^{m-1}$$

    $$T=111010010111 \quad ; P=101=5$$



  (note that the most significant digit is the rightmost one)
  - Find a fast way of computing t_{s+1} given t_s
  - $t_0=111=7$; $t_1=110=3$; $t_2=101=5$;
  - $t_3=100=1$; $t_4=010=2$; $t_5=100=1$; $t_6=001=4$;
  - Output all s such that t_s is equal to the number p represented by P

Warning

- In this lecture, *p* is for “pattern”, not for “prime”.
- All primes are denoted by *q*

The great formula

- How to transform
  $t_s=T[s+1]2^0+\underline{T[s+2]2^1+T[s+3]2^2+\ldots+T[s+m]2^{m-1}}$
  into
  $t_{s+1}= \underline{T[s+2]2^0+T[s+3]2^1+\ldots+T[s+m]2^{m-2}}+T[s+m+1]2^{m-1}$

e.g. T=111010010111–need to transform 111 =>110 => 101

- Three steps:
  - Subtract $T[s+1]2^0$
  - Divide by 2 (i.e., shift the bits by one position)
  - Add $T[s+m+1]2^{m-1}$
- Therefore: $t_{s+1}= (t_s- T[s+1]2^0)/2 + T[s+m+1]2^{m-1}$

Algorithm

- Can compute t_{s+1} from t_s using 3 arithmetic operations
- Therefore, we can compute all $t_0,t_1,\ldots,t_{n-m}$ using O(n) arithmetic operations
- We can compute a number corresponding to P using O(m) arithmetic operations
- Are we done ?

Problem

- To get O(n) time, we would need to perform each arithmetic operation in O(1) time
- However, the arguments are m-bit long !
- It is unreasonable to assume that operations on such big numbers can be done in O(1) time
- We need to reduce the number range to something more managable

Recall

- For any integers *a,b,q*
- (ab) mod q = ((a mod q) (b mod q)) mod q
- (a+b) mod q = ((a mod q) + (b mod q)) mod q

The great formula (revised)

- How to transform

$t'_s = (T[s+1]2^0 + \underline{T[s+2]2^1 + T[s+3]2^2 + \ldots + T[s+m]2^{m-1}}) \bmod q$

into

$t'_{s+1} = (\underline{T[s+2]2^0 + T[s+3]2^1 + \ldots + T[s+m]2^{m-2}} + T[s+m+1]2^{m-1}) \bmod q$

e.g. T=111010010111–need to transform 111 =>110 => 101

- Four steps:
  - Subtract $T[s+1]2^0$ (either 0 or 1)
  - Divide by 2 (i.e., shift the bits by one position)
  - Add $T[s+m+1](2^{m-1} \bmod q)$
  - Compute mod q of the result
- Therefore: $t'_{s+1} = \{(t'_s - T[s+1]2^0)/2 + T[s+m+1]2^{m-1} \} \bmod q$

Hashing

- We will instead compute

  $t'_s = T[s+1]2^0 + T[s+2]2^1 + \ldots + T[s+m]2^{m-1} \bmod q$

  where q is an "appropriate" prime number
- One can still compute t'_{s+1} from t'_s :

  $t'_{s+1} = (t'_s - T[s+1]2^0) * 2^{-1} + T[s+m+1]2^{m-1} \bmod q$
- If q is not large, i.e., has $O(\log n)$ bits, we can compute all t'_s (and p') in $O(n)$ time
- Recall $t'_s = t_s \bmod q$.
- Only if $t'_s = p \bmod q$ we check if $T^s = P$ (takes $O(m)$). Might be a false positive

Algorithm

- Let $\prod$ be a set of $2nm$ primes, each having $O(\log n)$ bits
- Choose q uniformly at random from $\prod$
- Compute t'_0, t'_1, …., and p'
- If $t'_s = p'$ check if $T[s+1 \ldots s+m-1] = P$ (might be a false positive.)

We will show that with high probability we have no false positive

False positives

- Consider any $t_s \neq p$. We know that both numbers are in the range $\{0 \ldots 2^m - 1\}$
- How many primes q are there such that

  $t_s \bmod q = p \bmod q$ that is,

  $(t_s - p) \bmod q = 0 \bmod q$

  $t_s - p = Kq$ for some integer K, and q is a divisor of $t_s - p$
- Such prime has to divide $x_s = t_s - p$
- Recall $x_s \leq 2^m$
- Represent $x = q_1^{e1} q_2^{e2} \ldots q_k^{ek}$, q_i prime, $e_i \geq 1$
- Since $2 \leq q_i$, we have $2^k \leq x_s \leq 2^m \rightarrow k \leq m$
- There are $\leq m$ primes dividing x_s

Analysis

- Call a prime q a **"bad prime"** for x_s if q divides t_s**-p** .
  that is, $t_s \bmod q = p \bmod q$ (false positive)

Lemma: q is a bad prime with probability $\leq ½$

- Let ∏ be a set of ***2nm*** primes, each having O(log n) bits
- We Choose q uniformly at random from ∏, and compute t'_0 , t'_1, …., t'_{n-m} and p'
- Pretend that we are crossing out from ∏ all the bad primes.
- Cross out the divisors of t_s-q, for s=0,1,2,…n-m
- At most ***mn*** primes are crossed out.
- So at least ***mn*** are left in ∏
- We picked q at random, so with probability $\geq ½$ it is **not** a bad prime. QED

Example m=2, n=4, |∏| =16
x_1=15, x_2=12, x_3=26 , x_4=49

∏={2 3 5 7 11 13 17 19
23 29 31 37 41 43 47 53}

Conclusion

- With probability $\geq ½$ ***we don't have any false positives***
- Also, the expected number of false positive is small, so the expected running time is O(n).

"Details"

- How do we know that such ∏ exists ?
- How do we choose a random prime from ∏ in O(n) time ?

Prime density

- Primes are "dense". I.e., if PRIMES(N) is the set of primes smaller than N, then asymptotically

  $$|PRIMES(N)|/N \sim 1/\log N$$

- If N large enough, then

  $$|PRIMES(N)| \geq N/(2\log N)$$

Prime density continued

- If we set $N=9mn \log n$, and N large enough, then

  $$|PRIMES(N)| \geq N/(2\log N) \geq 2mn$$

- All elements of PRIMES(N) are $\log N = O(\log n)$ bits long