CSc 445: Homework Assignment 1

Assigned: Wednesday Jan 23 2008,
Due: 10:30 AM, Monday Feb 4th 2008

Clear, neat and concise solutions are required in order to receive full credit so revise your work carefully before submission, and consider how your work is presented. If you cannot solve a particular problem, state this clearly in your write-up, and write down only what you know to be correct. For involved proofs, first outline the argument and the delve into the details.

1. (10 pts) Consider the problem of determining whether an arbitrary sequence $x_1, x_2, \ldots, x_n$ of n numbers contains repeated occurrences of some number.
   (a) Design an efficient algorithm for the case where you are not allowed to use additional space (i.e., you can use a few temporary variables, or $O(1)$ storage). What's the running time? Why?
   (b) Design a more efficient algorithms for the case where you're allowed to use additional memory (i.e., additional $O(n)$ storage). What's the running time? Why?
2. (10 pts) Prove by induction that the sum of the degrees of all the nodes in any graph is an even number.
3. (10 pts) Given two sets A and B, prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
4. (10 pts) Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions. Prove or disprove:
   (a) $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.
   (b) $f(n) + g(n) = \Theta(\min(f(n) + g(n)))$.
   (c) $f(n) = O(g(n))$ implies $g(n) = O(f(n))$.
   (d) $f(n) = \Theta(f(n/2))$.
5. (10 pts) Prove that no two consecutive numbers in the Fibonacci sequence have a common prime factor.
6. (10 pts) Determine and prove Θ bounds for the following recurrence relations:
   (a) $T(n) = 2T(n/2) + n \lg n$
   (b) $T(n) = T(n/2) + T(n/4) + T(n/8) + n$
   (c) $T(n) = 2T(n/2) + n/\lg n$
   (d) $T(n) = T(\lceil n/2 \rceil) + 7$
7. (10 pts) Use the recurrence tree to guess a good asymptotic bound for the following recurrence relations and then prove Θ bounds:
   (a) $T(n) = T(n-d) + T(d) + cn$, where and $c > 0$ and $d \geq 1$ are constants.
   (b) $T(n) = \begin{cases} c & \text{if } n < 3 \\ 3T(n/3) + cn & \text{if } n \geq 3 \end{cases}$, where $c > 0$
8. (10 pts) Consider the following program:

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Fib[n]
   if (n==0) then return 0
   else if (n==1) then return 1
   else return Fib[n-1]+Fib[n-2]
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   (a) What is the running time of this program?
   (b) Write pseudocode for a simple $O(n)$ algorithm that computes Fib(n).

9. (20 pts) Rank the following functions by order of growth, that is find an arrangement $g_1, g_2, \ldots, g_n$ so that $g_1 = \Omega(g_2), g_2 = \Omega(g_3), \ldots, g_{n-1} = \Omega(g_n)$, using asymptotic properties or by taking the limit. Each proof should be no more than 1-3 lines. Partition the set in equivalence classes, so that $g_i(n)$ and $g_j(n)$ are in the same equivalence class if and only if $g_i(n) = \Theta(g_j(n))$.

   $(\sqrt{2})^{\lg n}, n^0, n^1, n^{1.5}, n^2, n^3, n!, \lg n, \log_{10} n, (\lg n)!, 2^{\log_7 n}, \lg^2 n, e^n, 2^n, 2^{2^n}, n \lg n, 2^{lg^* n}, \phi^{n/\lg n}, (n+1)!, n^{\ln n}, \pi^n$

Extra Credit: During your latest visit to Mt. Lemmon, you noticed an long icicle, hanging off the ski lift. Enthralled by the powder, and possibly injured by the multiple face-plants, you forgot about it until the last run. You noticed that it had fallen down, broken into 3 pieces which formed a nice-looking isosceles triangle. On your way down the Catalina highway, you tried to calculate the probability that the tree pieces form a triangle (not just isosceles, but any triangle). 45 minutes later, you still were not sure what the right answer is. So, what is it?