Dating: Who wins the battle of the sexes?  
Stable marriage (matching) algorithm.

There are $n$ males and $n$ females  
Each female has her own ranked preference list of all the males  
E.g., women #1 most prefers male #3 over any other male.  
Each male has his own ranked preference list of the females  
How should we match them (1-to-1)
Definition of a Matching in this lecture

- A matching in this context is a list of couples that according to the algorithm, should be matched to each other. Each male is married to a single female and vice versa.
  \[ M = \{ (m_1, f_1), (m_2, f_2), \ldots, (m_n, f_n) \} \]

The algorithm aims to find a good matching (under some definition)

- Sometimes the term pairing is used

Rogue Couples

- Consider a given matching \( M \). Now suppose that some pair (male, female) who are not married to each other; actually prefer each other over their partners.
  - They will be called a rogue couple.
  - They both would gain from dumping their mates and marrying each other.

- A matching is called stable if it does not contains no rogue couples.

The study of stability will be the subject of the entire lecture.

We will: Analyze various mathematical properties of an algorithm that looks a lot like 1950's dating.
Given a set of preference lists, how do we find a stable pairing?

Wait! We don’t even know that such a pairing always exists!

Is there always a stable matching?

- Will show: every set of preference lists have a stable pairing.
- Will prove it by presenting a fast algorithm that, given any set of input lists, will output a stable pairing.

Terminology and principles

- A male can propose (marriage) to a female.
- A female can reject the proposal.

- During most of the process, a female would not accept a proposal, but would tell a proposing male "maybe":
- This is called "putting the male on a string"

- Once a male is rejected, he crosses off from his list the rejecting female – he will not propose to her again.
- Once a male proposes, he cannot change his mind until he is rejected.
**The Traditional Marriage Algorithm**

1. **Morning**
   - Each male to the best female whom he has not yet crossed off.

2. **Afternoon** (for each females with at least one proposal)
   - To today’s best offer: “Maybe, come back tomorrow” (putting him on a string)
   - All other proposals are rejected.

3. **Evening**
   - Any rejected male crosses the rejecting female off his list.

4. Until all males are on strings.

2) Each female marries the last male she just said “maybe”

**Note:** Each male proposes to females in decreasing order on his list.

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**Lemma:** If a female has a male $b$ on a string, then she will either marry him, or marry someone she prefers over him.

**Proof:**
- She would only let go of $b$ in order to “maybe” $b'$ which she prefers over $b$
- She would only let go of $b'$ for someone $b''$ she prefers over $b''$ etc.

When the process terminates, she is left with someone she prefers over $b$. 

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Corollary: Each female will marry her absolute favorite of the males who visit her during the Traditional Marriage Algorithm (TMA)

Lemma: No male can be rejected by all the females

- Proof by contradiction.
- Suppose male \( b \) is rejected by all the females. At that point:
  - Each female must have a suitor other than \( b \).
    (By previous Lemma, once a female has a suitor she will always have at least one)
  - The \( n \) females have \( n \) suitors, \( b \) not among them.
    Thus, there are at least \( n+1 \) males.
Contradiction

Theorem: The TMA always terminates after at most \( n^2 \) days

- Proof
  - The total length of the lists of all males is \( n \times n = n^2 \).
  - Each day at least one male gets a “No”, so at least one female is deleted from one of the lists.
  - Therefore, the number of days is bounded by the original size of the master list = \( n^2 \).
Great! We know that TMA will terminate and produce a pairing.

But is it stable?

**Theorem:** TMA. Produces a stable pairing.

1. Let \( m_i \) and \( f_i \) be any couple in \( T \).
2. Suppose \( m_i \) prefers \( f_2 \) over \( f_1 \).
3. We will argue that \( f_2 \) prefers her husband over \( m_i \).
4. During TMA, male \( m_i \) proposed to \( f_2 \) before he proposed to \( f_1 \).
5. Hence, at some point \( f_2 \) rejected \( m_i \) for someone she preferred.
6. By the Improvement lemma, the man she married was also preferable to \( m_i \).
7. Thus, every male will be rejected by any female he prefers to his wife.
8. \( T \) is stable. QED.

Forget TMA for a moment

- How should we define what we mean when we say “the optimal female for male \( b \)”?

**Flawed Attempt:**

“The female at the top of \( b \)’s list”
The Optimal female

• A male’s optimal female is the highest ranked female for whom there is some stable matching in which they are married.
• (note – this is not always the highest female on his list).
• She is the best female he can conceivably get in a stable world. Presumably, she might be better than the female he gets in the stable pairing output by TMA.

Thm

• The Traditional Marriage Algorithm yields a matching at which each male gets his optimal female

TMA but with exact clock.

Assume: At each time stamp, (every ‘tick’ of the clock) there is exactly one event:
• Event: a single man proposes, and if got rejected, his next proposal will be in next time stamp)

Note: The exact order is not crucial:
• If both \( m_1, m_2 \) are proposing to \( f \), the result is the same independent of whom proposed first.
**Thm:** TMA produces a male-optimal pairing

- **Proof:** Suppose, for a contradiction, that some male gets rejected by his optimal female during TMA.
- Let $t$ be the earliest time at which a male (Adam) got rejected by his optimal female $f$ (Florence).
- Florence rejected Adam because she said "maybe" a preferred male $m_2$ (BOB).
- Bob had not yet been rejected by his optimal female (by the definition of $t$). Therefore In Bob's list
  - Florence is either the optimal female of Bob. Or Florence is higher than his optimal.
That is, in any stable world, BOB would either be married to Florence.
- Let $S$ be the matching at which (Adam, Florence) are married
  - $S$ is NOT the result of the TMA
- (Bob, Florence) are a rouge couple. QED

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**The Pessimal male**

- Let $f$ (Florence) be one of the females.
- Note that there might be different matching which are stable.
  - Florence's pesimal male is the lowest ranked male (on her list) for whom there is some stable matching at which the female gets him.
  - He is the worst male she can conceivably get in a stable world.

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**Thm:** The TMA is female-pessimal.

- **Proof:** We know TMA it is male-optimal.
  - (Syndrome, Florence) is a couple in TMA,
  - $\Rightarrow$ Florence is Syndrome's optimal female.
Suppose there is a stable pairing $S$ where some female $f$, Florence does worse than in TMA.
  - Let Syndrome be Florance's husband in TMA.
  - Let Zod be Florance's husband in $S$
    - (Zod is lower on her list than Syndrome)
  - By assumption, Syndrome prefers Florence over his wife in $S$
    - (since Florence is his optimal female)
  - So (Syndrome, Florence) is a rogue couple.
  - Therefore, $S$ is not stable. QED
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