

This homework is due Thursday November 16 at the start of class. Questions are drawn from the material in Sections 2.3, 3.1, and 3.2 of the text on non-context-free languages and Turing machines.

The homework is worth 100 points. When questions with several parts do not specify the points for each part, each part has equal weight.

Remember to write on just one side of a page, do not use scrap paper, put your answers in the correct order, and staple your pages together. If you can't solve a problem, state this, and write only what you know to be correct. Neatness and conciseness count.

- (1) **(Deciding context-freeness)** (30 points) Prove or disprove that the following languages are context-free.

(a) $\{0^i 1^j 2^k : k > \max\{i, j\} \text{ where } i, j, k \geq 0\}$

(b) $\{0^i 1^j 2^k : k > i + j \text{ where } i, j, k \geq 0\}$

(c) $\{0^i 1^j 2^k : k > ij \text{ where } i, j, k \geq 0\}$

(Hint: To *disprove* that a language is context-free, use the Pumping Lemma. To *prove* that a language is context-free, give a pushdown automaton or a context-free grammar for the language.)

- (2) **(Closure properties of context-free languages)** (30 points)

- (a) Prove that context-free languages are closed under **reversal**. More formally, prove that if L is context-free, then

$$\text{Reverse}(L) := \{w^R : w \in L\}$$

is also context-free.

- (b) Prove that context-free languages are closed under taking all **substrings**. More formally, prove that if L is context-free, then

$$\text{Substring}(L) := \{y : xyz \in L \text{ where } x, y, z \in \Sigma^*\}$$

is also context-free. Here Σ is the alphabet for language L .

- (c) Prove that context-free languages are *not* closed under **complementation**. More formally, prove that if L is context-free, then

$$\text{Complement}(L) := \{w \in \Sigma^* : w \notin L\}$$

is not necessarily context-free. Here Σ is the alphabet for language L .

(Hint: In Part (c), it may be easier to find a specific language L such that L is *not* context-free but $\text{Complement}(L)$ is context-free. One such candidate for L is $\{a^n b^n c^n : n \geq 0\}$. In Parts (a) through (c), to prove that a language is context-free, construct a pushdown automaton or a context-free grammar for the language.)

- (3) **(Designing Turing machines)** (10 points) In Turing machine computations, it is often useful to have a special symbol that marks the left end of the tape (just as with pushdown automata, it is often useful to have a bottom-of-stack marker).

Give a *transition diagram* for a single-tape Turing machine that places the symbol \$ in the first cell on the tape and shifts the entire initial contents of the tape one cell to the right. Once your Turing machine has completed this task, it should enter its accept state.

You may assume that the input to the Turing machine is over alphabet Σ and that symbol \$ does not appear in Σ .

- (4) **(Closure properties of decidable languages)** (30 points) Prove that the class of decidable languages is closed under the following operations:

- (a) reversal,
- (b) extending with an arbitrary prefix and suffix, and
- (c) complementation,

where the operations in Parts (a) and (c) are defined in the corresponding parts of Problem (2). The operation in Part (b) above on a language L is more formally

$$\text{Extend}(L) := \{xyz : y \in L \text{ where } x, y, z \in \Sigma^*\}.$$

In your proofs, describe a Turing machine by giving a *high-level description* (in other words, do not give a low-level transition diagram). You may use a *multitape* Turing machine to simplify your solution, but your machine must be *deterministic*.

- (5) **(Bonus) (Moving a head)** (10 points) With multitape Turing machines as defined in the book, a single transition moves *all* heads on all tapes simultaneously and independently. Suppose instead you want to move just *one* head in a multitape Turing machine. Show how to simulate the ability to move a single head left or right by one cell, in a multitape Turing machine that obeys the books definition.

(Hint: Before moving the desired head in the given direction, it may be useful to move all heads to the left ends of their tapes.)

Note that Problem (5) is a bonus question. It is not required (except for the honors section) and its points are not added to regular points.