CSc/Math 473 **homework 1** **August 27, 2007**

DUE: Tuesday 11 September in class

Reading
Text, Chapter 1: Regular Languages
Problems
Clarity and brevity of answers wins points, so revise your work before writing it up for submission. Begin each problem on a separate sheet, state the problem (briefly), and please *write on only one side* of each sheet. Submit problems in the labeled envelopes that will be provided in class.

1. Let $R = \{(a,c),(c,e),(e,e),(e,b),(d,b),(d,d)\}$. Draw directed graphs representing each of the following:

   (a) R

   (b) R^2

   (c) R^{-1}

   (d) $R \cap R^2$

2. Draw directed graphs representing relations of the following types:

   (a) Reflexive, transitive and *antisymmetric.* Antisymmetry of R means that $(\forall a, b)$ if $(a,b) \in R$ and $a \neq b$ then $(b,a) \notin R$.

   (b) Reflexitve, transitive, and *neither* symmetric nor antisymmetric.

3. What is the the reflexive, transitive closure R^* of $R = \{(a,b),(a,d),(d,c),(d,e)\}$? Express R^* as a set, and draw a directed graph representing R^*.

4. Text, Problem 0.10. Identify *exactly* the step in the argument that is wrong.

5. Show *by induction* that $n^4 - 4n^2$ is divisible by 3 for all $n \geq 0$. *HINT:* Factor $i^4 - 4i^2$. Then 3 must divide at least one of the factors.

6. (*Ackermann's Function*) Consider the following recursively (inductively) defined function, where m, n are non-negative integers.

$$\begin{aligned} A(n,0) &= n+1 \\ A(0,m+1) &= A(1,m) \\ A(n+1,m+1) &= A(A(n,m+1),m) \end{aligned}$$

   Find simple expressions for the following functions (use inference and experiment to make a discovery), and then *prove* by induction that your expression is correct (use deduction to confirm your discovery).

   (a) $A(n,1)$

   (b) $A(n,2)$

   (c) $A(n,3)$

   Caution: If, as part of the discovery process, you write a program to explore this function's behavior, be sure to use only small values of the argument n, or we could be waiting forever for you to return. Especially in part (c).

7. The *reversal* of a string w, denoted w^R, is the string "spelled backwards". For example $(reverse)^R = esrever$. A careful definition can be given by induction on the length of a string: If w is a string of length 0, then $w^R = \epsilon$. If $w = ua$ is a string of length $n+1 > 0$, where $a \in Sigma$, and $u \in Sigma^*$, then $w^R = au^R$.

   Prove the following by induction:

   (a) $(xy)^R = y^R x^R$

   (b) $(w^R)^R = w$. *HINT:* Use (a).

8. Point out the flaw in the reasoning below. Find the precise step or steps that are in error, and give a concrete (counter-)example that shows this step is in error. It is recommended that in debugging this proof that you try stepping through the argument with small sets of marbles [1].

   Theorem: All marbles have the same color.

   Proof: Let A be any set of n marbles, where $n \geq 1$. The proof is by induction on n.

   Base: If $n = 1$ all marbles in A clearly have the same color.

   Step: Assume that if A is any set of k marbles, then all marbles in A have the same color. Let A' be a set of $k+1$ marbles, where $k \geq 1$. Remove one marble m from A'. We are then left with a set A'' of k marbles. By the induction hypothesis, A'' has all marbles of the same color. Remove from A'' a second marble m', and then add back to A'' the marble m originally removed. We again have a set of k marbles, which by the induction hypothesis has marbles all the same color. Thus the two marbles m, m' removed must have been the same color, and so the set A' must contain marbles all the same color.

   By the principle of induction, applied to the above basis and step, we conclude that in any set of $n \geq 1$ marbles, all marbles are the same color. **QED**.

9. Text, Problem 0.12. By "graph" is here meant "*un*directed graph". This theorem is often stated in a more humane manner: "In any group of 2 or more people, there are at least two persons that have the same number of acquaintances within the group." In your argument, make use of the *Pigeonhole Principle*: if p pigeons are placed in fewer than p pigeonholes, then at least one pigeonhole must have more than one pigeon in it.

10. Show each of the following:

    (a) $\{\epsilon\}^* = \{\epsilon\}$.

    (b) For any language L, $L^*L^* = L^*$.

    (c) For any language L, $(L^*)^* = L^*$.

    Remember that to show two sets A and B are the same, you have two things to do: (1) show $A \subseteq B$ and (2) show $B \subseteq A$.

11. In the *Scheme* programming language[2], the syntactic category of "Numbers" is (partially) defined as follows[3]:

<num>	$\rightarrow$	<prefix><real>
<real>	$\rightarrow$	<sign><ureal>
<ureal>	$\rightarrow$	<uint> | <uint>/<uint> | <decimal>
<uint>	$\rightarrow$	<digit>$^+$
<prefix>	$\rightarrow$	ϵ | #d
<digit>	$\rightarrow$	0 | ... | 9
<decimal>	$\rightarrow$	<uint><exp> | .<digit>$^+$<suffix> | <digit>$^+$.<digit>*<suffix>
<suffix>	$\rightarrow$	ϵ | <exp>
<exp>	$\rightarrow$	<marker><sign><digit>$^+$
<sign>	$\rightarrow$	ϵ | + | -
<marker>	$\rightarrow$	e | s | f | d | l

    (The markers `s, f, d, l` override the default floating point `e` representation, in implementations that support short, single precision, double and long.)

    Problem: Draw the state diagram that recognizes a well-formed `<num>` in Scheme. Your machine can be non-deterministic. To save space, label edges—where possible—with a *class* of symbols, such as `<digit>`, rather than introducing an edge for each symbol.

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[1] Take care not to lose your marbles.

[2] Dybvig, R.K., *The Scheme Programming Language, 3rd Ed.*, Cambridge: The MIT Press, 2003.

[3] Scheme also provides for numbers in radix 2, 8 and 16. In addition, Scheme employs a special "digit" called # to indicate "inexact" digits, and calculations must then report on resulting digits that are inexact given the information provided. For simplicity, we have suppressed everything except the rules for *decimal exact numbers* in this exercise.