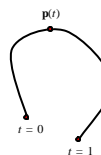


Bezier Curves

CS 319
Advanced Topics in Computer Graphics
John C. Hart

Parametric Curves

- Function \mathbf{p} of single parameter t
 - $\mathbf{p}(t) = (t, x(t))$ (explicit) function curve
 - $\mathbf{p}(t) = (x(t), y(t))$ plane curve
 - $\mathbf{p}(t) = (x(t), y(t), z(t))$ space curve
- Tangent
 - $\mathbf{p}'(t)$ points in tangent direction
 - $\mathbf{p}'(t) = (x'(t), y'(t), z'(t))$
 - $\mathbf{p}'(t) / \|\mathbf{p}'(t)\|$ - tangent direction
 - $\|\mathbf{p}'(t)\|$ - parametric velocity



Arc Length

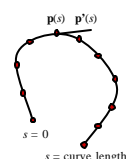
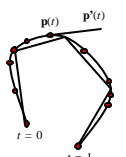
- Arc length parameterization

$$s(t) = \int_0^t \sqrt{\mathbf{p}'(\mathbf{t}) \cdot \mathbf{p}'(\mathbf{t})} dt$$

$$= \int_0^t \sqrt{x'^2(\mathbf{t}) + y'^2(\mathbf{t}) + z'^2(\mathbf{t})} dt$$

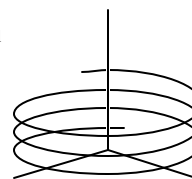
- Parameter s defined as parameter t where length of curve from $\mathbf{p}(0)$ to $\mathbf{p}(t)$ equals s

- Velocity $\|\mathbf{p}'(s)\| = 1$



Arc Length Example

- Circular helix
 - $\mathbf{p}(t) = (r \cos t, r \sin t, ct)$
 - Parameter r controls radius of spiral
 - Parameter c controls how quickly \mathbf{p} winds around the z axis
- Tangent
 - $\mathbf{p}'(t) = (-r \sin t, r \cos t, c)$
- Square
 - $\mathbf{p}'(t) \cdot \mathbf{p}'(t) = (r^2 + c^2)$
- Integrate
 - $s(t) = t(r^2 + c^2)^{1/2}$



Curvature

- Extrinsic Curvature

- Arc length
 - $\mathbf{k} = \|\mathbf{p}''(s)\|$

- Plane curve

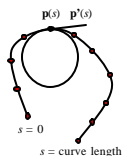
$$\mathbf{k} = (x'(t)y''(t) - y'(t)x''(t)) / (x'^2(t) + y'^2(t))^{3/2}$$

- Space curve

$$\mathbf{k} = \|\mathbf{p}'(t) \times \mathbf{p}''(t)\| / \|\mathbf{p}'(t)\|^3$$

- Osculating circle

- Radius of curvature $\mathbf{r}(s) = 1/\mathbf{k}(s)$
- Curvature units: $1/m$



Torsion

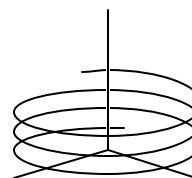
- Triple Scalar Product
 - $\mathbf{t}(s) = \mathbf{r}(s) \mathbf{p}'(s) \cdot (\mathbf{p}''(s) \times \mathbf{p}'''(s))$
 - $= \mathbf{r}(s) \|\mathbf{p}'(s) \mathbf{p}''(s) \mathbf{p}'''(s)\|$
- Positive for "right handed" curves
- Negative for "left handed" curves

Triple Scalar Product

$$|\mathbf{abc}| = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

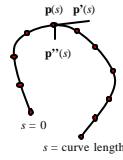
$$= |\mathbf{bca}| = |\mathbf{cab}|$$

$$= \begin{vmatrix} x_a & x_b & x_c \\ y_a & y_b & y_c \\ z_a & z_b & z_c \end{vmatrix}$$



Frenet Frame

- Tangent: $T(s) = \mathbf{p}'(s)$
 - Already unit length since parameterized by arc length
- Normal: $N(s) = \mathbf{p}''(s) / \|\mathbf{p}''(s)\|$
 - Normal points "inside" curve
 - Also $\mathbf{p}''(s) / \kappa(s)$ or $\kappa(s) \mathbf{p}''(s)$
- Binormal: $B(s) = T(s) \times N(s)$
- Frenet Frame: $(T(s), N(s), B(s))$



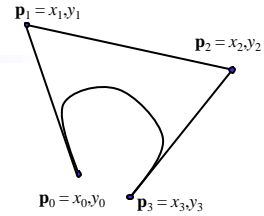
Frenet Formulas

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & t \\ 0 & -t & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

Bezier Curves

$$\mathbf{p}(t) = \sum_{i=0,3} B_i(t) \mathbf{p}_i$$

$$B_i(t) = \binom{3}{i} t^i (1-t)^{3-i}$$



$$\mathbf{p}(t) = (1-t)^3 \mathbf{p}_0 + 3(1-t)^2 t \mathbf{p}_1 + 3(1-t) t^2 \mathbf{p}_2 + t^3 \mathbf{p}_3$$

$$\begin{aligned} x(t) &= (1-t)^3 x_0 + 3(1-t)^2 t x_1 + 3(1-t) t^2 x_2 + t^3 x_3 \\ y(t) &= (1-t)^3 y_0 + 3(1-t)^2 t y_1 + 3(1-t) t^2 y_2 + t^3 y_3 \end{aligned}$$

coordinate system dependent!

Bernstein Polynomials

- Defined for any degree

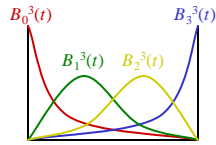
$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$
- n choose i

$$\binom{n}{i} = n! / (i!(n-i)!) = \binom{n-1}{i} + \binom{n-1}{i-1}$$
- Partition of unity
 - Sum to one for any t in $[0,1]$

$$\sum_{i=0}^n B_i^n(t) = 1$$
- Higher degrees lerps of lower degrees

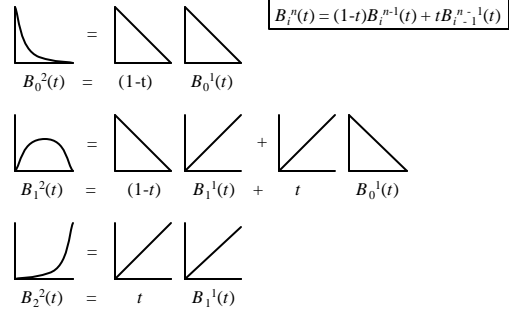
$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} = \binom{n-1}{i} t^i (1-t)^{n-i} + \binom{n-1}{i-1} t^{i-1} (1-t)^{n-i+1}$$

$$= (1-t) B_i^{n-1}(t) + t B_{i-1}^{n-1}(t)$$



Hint: Multiply left term by $(n-i)/(n-i)$

Building Bernsteins



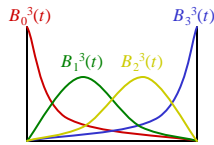
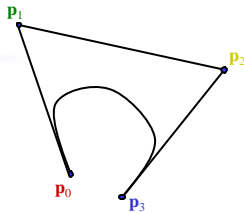
Bezier Properties

- Affine invariance
 - Barycentric combination
- Control points form convex hull
 - Bernstein basis in $[0,1]$
- Endpoint interpolation

$$B_0^n(0) = 1; B_n^n(1) = 1$$
- Symmetry

$$B_i^n(t) = B_{n-i}^n(1-t)$$
- Derivatives

$$B_i^n(t) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$$



de Casteljau Algorithm

- Cascading lerps

$$\mathbf{p}_{01} = (1-t) \mathbf{p}_0 + t \mathbf{p}_1$$

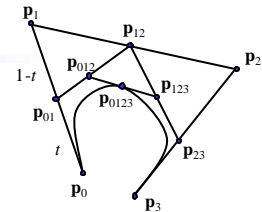
$$\mathbf{p}_{12} = (1-t) \mathbf{p}_1 + t \mathbf{p}_2$$

$$\mathbf{p}_{23} = (1-t) \mathbf{p}_2 + t \mathbf{p}_3$$

$$\mathbf{p}_{012} = (1-t) \mathbf{p}_{01} + t \mathbf{p}_{12}$$

$$\mathbf{p}_{123} = (1-t) \mathbf{p}_{12} + t \mathbf{p}_{23}$$

$$\mathbf{p}_{0123} = (1-t) \mathbf{p}_{012} + t \mathbf{p}_{123}$$
- Subdivides curve at \mathbf{p}_{0123}
 - $\mathbf{p}_0 \mathbf{p}_{01} \mathbf{p}_{012} \mathbf{p}_{0123}$
 - $\mathbf{p}_{0123} \mathbf{p}_{123} \mathbf{p}_{23} \mathbf{p}_3$
- Repeated subdivision converges to curve



coordinate free!

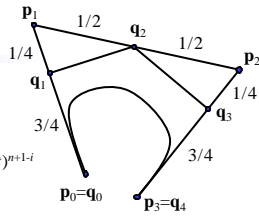
Degree Elevation

- Used to add more control over a curve
- Start with $\sum \mathbf{p}_i \binom{n}{i} t^i (1-t)^{n-i} = \sum \mathbf{q}_i \binom{n+1}{i} t^i (1-t)^{n+1-i}$
- Now figure out the \mathbf{q}_i

$$(t+(1-t)) \sum \mathbf{p}_i \binom{n}{i} t^i (1-t)^{n-i} = \sum \mathbf{p}_i \binom{n}{i} t^i (1-t)^{n+1-i} + \sum \mathbf{p}_i \binom{n}{i} t^{i+1} (1-t)^{n-i}$$
- Compare coefficients

$$\mathbf{q}_i \binom{n+1}{i} = \mathbf{p}_i \binom{n}{i} + \mathbf{p}_{i-1} \binom{n}{i-1}$$

$$\mathbf{q}_i = (i/(n+1))\mathbf{p}_{i-1} + (n+1-i)/(n+1)\mathbf{p}_i$$
- Repeated elevation converges to curve



Blossoms

- Defines a *polar form* $f(\dots)$ for a polynomial $F(t)$
- Number of blossom parameters = degree of polynomial (n)
- Symmetric
 - $f(t_1, t_2, t_3) = f(t_2, t_1, t_3) = f(t_3, t_2, t_1)$
- Multiaffine
 - $f(\dots, \sum a_i t_i, \dots) = \sum a_i f(\dots, t_i, \dots)$
- Evaluation by *diagonalization*
 - $F(t) = f(t, t, \dots, t)$
 - $F^{(q)}(t) = n!(n-q)! f(t, t, \dots, t, \underbrace{1, \dots, 1}_q)$

Bezier Blossoming

$\mathbf{p}(t) = f(t,t,t) = (1-t)f(t,t,0) + t f(t,t,1)$

$= (1-t)[(1-t)f(t,0,0) + t f(t,0,1)] + t [(1-t)f(t,0,1) + t f(t,1,1)]$

$= (1-t)^2 f(t,0,0) + 2(1-t)t f(t,0,1) + t^2 f(t,1,1)$

$= (1-t)^3 f(0,0,0) + 3(1-t)^2 t f(0,0,1) + 3(1-t)t^2 f(0,1,1) + t^3 f(1,1,1)$

Systolic Array

