This homework is due Monday, September 22, at the start of class. The questions are drawn from Chapters 3 and 4 of the text and from the material given in class on asymptotics, sums, recurrences, and divide and conquer.

The homework is worth a total of 100 points. When questions with several parts do not specify the points for each part, each part has equal weight.

Remember to write on just one side of a page, do not use scrap paper, put your answers in the correct order, and staple your pages together. If you can't solve a problem, state this, and write only what you know to be correct. Neatness and conciseness count.

(1) **(Ordering functions asymptotically)** (20 points)

(a) (20 points) Order the following functions according to their rate of growth. Specifically, group the functions into equivalence classes such that functions f and g are in the same class iff $f = \Theta(g)$, and then order the equivalence classes from slowest to fastest growing.

For each successive pair of functions (f, g) in your order, prove the relationship between f and g (either $f = \Theta(g)$ or $f = o(g)$) using the list of asymptotic properties given in class or possibly by taking a limit. Each proof should be no more than one to three lines, and should give the name of the property used in each step.

Organize your write-up so that you (1) first give the ordered list of functions, with a footnote labeling each successive pair in the list, and (2) then separately give the short proof for each pair in a list of footnotes.

$(\frac{3}{2})^n$	n^2	$n!$	$\lfloor \ln n \rfloor !$
n	$n \ln n$	$\ln(n!)$	2^{2^n}
$n2^n$	$n^{\ln \ln n}$	$\ln n$	$2^{2^{n+1}}$
2^n	$(\ln n)^{\ln n}$	$(n+1)!$	$\ln \ln n$

(Hint: First form a rough initial order of the functions, and then use insertion sort on this list to obtain the final order.)

(b) **(bonus)** (5 points) Prove or disprove the following conjecture.

Conjecture *Let $f(n)$ be an asymptotically positive function. Suppose $f(n) = \omega(n^c)$ for every c. Then $f(n) = \Omega(b^n)$ for some b.*

(Note: The above conjecture says that if f is growing faster than a polynomial, then it is growing exponentially fast.)

(2) **(Summing asymptotics)** (20 points)

(a) (20 points) Prove the following.

Theorem *Let $f(n)$ be a function for which*

$$\sum_{1 \le i \le n} f(i) \;=\; \omega(1).$$

Then

$$\sum_{1\le i\le n} \Theta\big(f(i)\big) \;=\; \Theta\Big(\sum_{1\le i\le n} f(i)\Big).$$

According to our interpretation of asymptotic expressions, this says the following. Let $g(n)$ be a function such that $g \in \Theta(f)$, and let $h(n) = \sum_{1\le i\le n} f(i)$. Then $\sum_{1\le i\le n} g(i) \in \Theta(h)$.

(b) **(bonus)** (5 points) Disprove the following.

Conjecture *Let $f(n)$ be an asymptotically positive function. Then*

$$\sum_{1\le i\le n} \Theta\big(f(i)\big) \;=\; \Theta\Big(\sum_{1\le i\le n} f(i)\Big).$$

(Note this is Part (a) with the condition on f relaxed.)

(3) **(Solving recurrences)** (20 points) Derive a Θ-bound on the solution to each of the following recurrences. Use the Master Theorem (including the specializations given in class), the iteration method, or the substitution method. Do *not* use the change-of-variable technique from the end of Section 4.1, or the recursion tree method from Section 4.2.

Do not worry about taking floors or ceilings of arguments.

(a) $T(n) \;=\; 4T(n/2) \;+\; n^2\sqrt{n}.$

(b) $T(n) \;=\; 3T(n/2) \;+\; n\lg n.$

(c) $T(n) \;=\; \sqrt{n}T(\sqrt{n}) \;+\; n.$

(d) $T(n) \;=\; T(n/2) \;+\; T(n/4) \;+\; T(n/8) \;+\; n.$

(4) **(Finding the missing integer)** (20 points) Suppose you have a 2-dimensional array $A[1\!:\!n, 1\!:\!k]$ of bits, where $k = \Theta(\log n)$. Each row of A represents an integer in the range $[0, n]$, written in binary. All integers 0, 1, …, n are represented by the rows of A, *except one*, which is missing.

Design an algorithm that finds the missing integer in $\Theta(n)$ time, assuming the only operation you can perform on A is to read a bit $A[i, j]$. Reading a bit takes $\Theta(1)$ time.

(Note: Keep in mind that copying a row of A will take $\Theta(\log n)$ time.)

(5) **(Maximum weight interval)** (20 points) In the Maximum Weight Interval Problem, you are given an array $A[1\!:\!n]$ of positive and negative numbers, and you are asked to find an interval $[i, j]$ of the array, where $1 \le i \le j \le n$, such that the sum of the elements in $A[i:j]$, namely $\sum_{i\le k\le j} A[k]$, is maximum over all possible intervals.

Design a divide-and-conquer algorithm for this problem that runs in $\Theta(n\log n)$ time.

(Note: There are $\Omega(n^2)$ possible intervals, so an $O(n\log n)$ time algorithm must find an optimal interval without examining all intervals.)

(Hint: Organize the search for the maximum weight interval by dividing the array in half; the optimum interval lies in the left half, the right half, or spans the midpoint.)

Note that Problems (1)(b) and (2)(b) are a bonus questions. They are not required, and their points are not added to regular points.