CS 545

Dynamic Programming

Slides courtesy of Charles Leiserson with small changes by Carola Wenk

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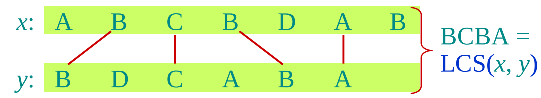
Dynamic programming

Example 1: ***Longest Common Subsequence (LCS)***

- Given two sequences $x[1\,.\,.\,m]$ and $y[1\,.\,.\,n]$, find a longest subsequence common to them both.

"a" *not* "the"



Different phrasing: Find a set of a maximum number of segments, such that
•Each segment connects a character of x to an identical character of y,
•Each character is used at most once
•Segments do not intersect.

Brute-force LCS algorithm

Check every subsequence of $x[1\,.\,.\,m]$ to see if it is also a subsequence of $y[1\,.\,.\,n]$.

Analysis

- Checking = $\Theta(m+n)$ time per subsequence.
- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).

Worst-case running time = $\Theta((m+n)2^m)$
= exponential time.

Towards a better algorithm

Simplification:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

Notation: Denote the length of a sequence s by $|s|$.

Strategy: Consider ***prefixes*** of x and y.

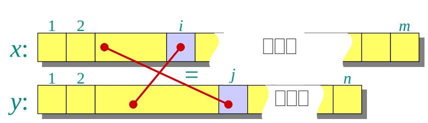
- Define $c[i, j] = |\mathrm{LCS}(x[1\,.\,.\,i], y[1\,.\,.\,j])|$.
- Then, $c[m, n] = |\mathrm{LCS}(x, y)|$.

Recursive formulation

Theorem.

$$c[i,j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

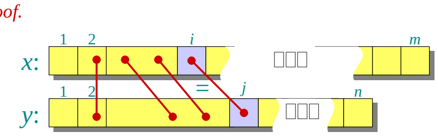
Proof: First Note that it is impossible that
x[i] is matched to an element in *y[1..j-1]* and in addition
y[j] is matched to an element in *x[1..i-1]*



Recursive formulation-cont

Case (I): $x[i] = y[j]$. Claim: $c[i, j]=c[i-1,j-1]+1$.

Proof.



We claim that there is a max matching that matches $x[i]$ to $y[j]$.

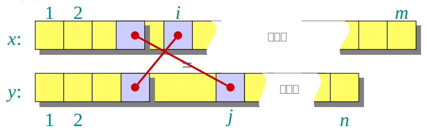
Indeed, if $x[i]$ is matched to $y[k]$ (for $k<j$) then $y[j]$ is unmatched (otherwise we have two crossing segments). Hence we can obtain another matching of the same cardinality by match $x[i]$ to $y[j]$.

This implies that we can match $x[1..i-1]$ to $y[1..j-1]$, and add the match $(x[i],y[j])$. So $c[i, j]=c[i-1,j-1]+1$

Recursive formulation-cont

Case (II): $x[i] \neq y[j]$ Claim: $c[i, j]=\max\{c[i-1, j], c[i, j-1]\}$

Recall - in LCS($x[1..i]$, $y[1..j]$) it cannot be that **both** $x[i]$ and $y[j]$ are both matched.



If $x[i]$ is unmatched then
LCS($x[1..\mathbf{i}]$, $y[1..\mathbf{j}]$)= LCS($x[1..\mathbf{i-1}]$, $y[1..\mathbf{j}]$)

If $y[j]$ is unmatched then
LCS($x[1..\mathbf{i}]$, $y[1..\mathbf{j}]$)= LCS($x[1..\mathbf{i}]$, $y[1..\mathbf{j-1}]$)

So $c[i, j]= \max\{c[i-1, j], c[i, j-1]\}$

Dynamic-programming hallmark #1

Optimal substructure
An optimal solution to a problem (instance) contains optimal solutions to subproblems.

If z = LCS(x, y), then any prefix of z is an LCS of a prefix of x and a prefix of y.

Recursive algorithm for LCS

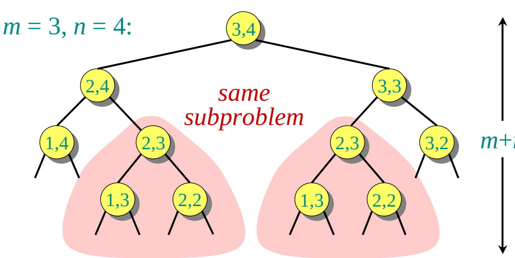
LCS(x, y, i, j)
  if (i==0 **or** j=0) return 0
  if $x[i] = y[j]$
    then return LCS(x, y, $i-1$, $j-1$) + 1
    else return $\max\{$ LCS(x, y, $i-1$, j), LCS(x, y, i, $j-1$)$\}$

To call the function LCS(x, y, m,n)

Worst-case: $x[i] \neq y[j]$, for all i,j in which case the algorithm evaluates two subproblems, each with only one parameter decremented.

Recursion tree

$m = 3$, $n = 4$:



Height = $m + n \Rightarrow$ work potentially 2^{m+n} exponential.
but we're solving subproblems already solved!

Dynamic-programming hallmark #2

Overlapping subproblems
A recursive solution contains a "small" number of distinct subproblems repeated many times.

The number of distinct LCS subproblems for two strings of lengths m and n is only mn.

Memoization algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

LCS(x, y)
  for i=0 **to** m $c[i, 0] = 0$
  for j=0 **to** n $c[0, j] = 0$

  for i=1 **to** m
   for j=1 **to** n
    if ($x[i] = y[j]$)
     then $c[i, j] \leftarrow c[i-1, j-1] + 1$
     else $c[i, j] \leftarrow \max\{ c[i-1, j], c[i, j-1] \}$

Time = $\Theta(mn)$ = constant work per table entry.
Space = $\Theta(mn)$.

LCS: Dynamic-programming algorithm

LCS(X,Y)="BCBA"

X=B D C A B A

Y=A B C B D A B

X \ Y=		1 A	2 B	3 C	4 B	5 D	6 A	7 B
	0	0	0	0	0	0	0	0
1 B	0	0	1	1	1	1	1	1
2 D	0	0	1	1	1	2	2	2
3 C	0	0	1	2	2	2	2	2
4 A	0	1	1	2	2	2	3	3
5 B	0	1	2	2	3	3	3	4
6 A	0	1	2	2	3	3	4	4

Reconstruction *z=LCS(x,y)*

IDEA: Compute the table bottom-up. Fill *z* backward.

LCS(x,y)="BCBA"

x=B D C A B A

y=A B C B D A B

Observation: *c[i;j]≥c[i-1;j]* and *c[i;j] ≥c[i;j-1]*
Proof Sketch: We use a longer prefix, so there are more chars to be match.

LCS Reconstruction:
Set *i=m; j=n; k=c[i;j]*
While(*k*>*0*){
  if (*c[i;j]*>*c[i-1;j]* and *c[i;j]*>*c[i;j-1]*) {
    z[k] = x[i] ;
    i--; j-- ; k-- ;
  }else /**c[i;j]=c[i-1;j]* or *c[i;j]=c[i-1;j]* */
  if (*c[i;j]*==*c[i;j-1]*) *j-- ;*
  else *i--* ;
}

		1 A	2 B	3 C	4 B	5 D	6 A	7 B
	0	0	0	0	0	0	0	0
1 B	0	0	1	1	1	1	1	1
2 D	0	0	1	1	1	2	2	2
3 C	0	0	1	2	2	2	2	2
4 A	0	1	1	2	2	2	3	3
5 B	0	1	2	2	3	3	3	4
6 A	0	1	2	2	3	3	4	4

Reconstructing *z=LCS(X,Y)*

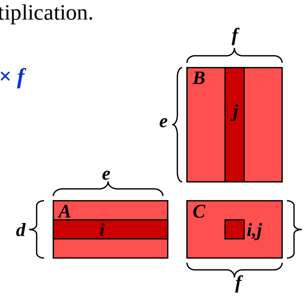
Another idea – While filling *c[],* add arrows to each cell *c[i,j]* specifying which neighboring cell *c[i,j]* it got its value.
• *c[i,j].flag* = "\ " if *c[i,,j]=c[i-1;j-1]+1*
• *c[i,j].flag* = "↑ " if *c[i,,j]=c[i-1;j]*
•*c[i,j].flag* = "←" if *c[i,,j]=c[i-1;j]*

		A	B	C	B	D	A	B
	0	←0	0	0	0	0	0	0
B	0	↑0	1	←1	1	1	1	1
D	0	↑0	↑1	←1	←1	2	2	2
C	0	↑0	↑1	2	←2	←2	2	2
A	0	1	↑1	↑2	←2	2	3	3
B	0	↑1	2	←2	3	←3	3	4
A	0	1	↑2	←2	↑3	←3	4	←4

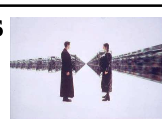
Example 2 of dynamic programming: **Matrix Chain-Products**

- Review: Matrix Multiplication.
  - ***C = AB***
  - ***A*** is ***d × e***, ***B*** is ***e × f***
  - ***O(def***) time

$$C[i,\ j] = \sum_{k=0}^{e-1} A[i,k]\ B[k,\ j]$$



Matrix Chain-Products



- **Matrix Chain-Product:**
  - Compute $A = A_0 A_1 \ldots A_{n-1}$
  - A_i is $d_i \times d_{i+1}$
  - Problem: How to **parenthesize?**
- Example 1. $(A_1 A_2)(A_3 A_4) = A_1 (A_2 (A_3 A_4)) =$
  $(A_1 (A_2 A_3)) A_4 = A_1((A_2 A_3) A_4) = \ldots$
- Example 2
  - *B* is 3 × 100
  - *C* is 100 × 7
  - *D* is 7 × 5
  - *(BC)D* 3×100×7 + 7×5×5 = 2275 mults
  - *B(CD)* 3×100×5 + 100×7×5 = 5000 mults

An Enumeration Approach

- **Matrix Chain-Product Alg.:**
  - Try all possible ways to parenthesize $A=A_0 A_1 \ldots A_{n-1}$
  - Calculate number of ops for each one
  - Pick the one that is best
- Running time:
  - # of parenthesizations = # of binary trees with n nodes
    - **Exponential**!
    - Called the n^{th} Catalan number – it is almost 4^n.
  - This is a terrible algorithm!

A Greedy Approach

Repeatedly select the product that uses the fewest operations.

Counter-example:

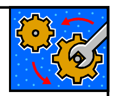
- A is 101 × 11
- B is 11 × 9
- C is 9 × 100
- D is 100 × 99

- Idea selects A((BC)D) 109989+9900+108900=228789 mults
- Best is (AB)(CD)
  9999+89991+89100=189090 mults

A "Recursive" Approach

- Define **subproblems**:
  - Find the best parenthesization of $A_i A_{i+1} \ldots A_j$.
  - Let $N_{i,j}$ = # of operations done by this subproblem.
  - The optimal solution for the whole problem is $N_{0,n-1}$.
- **Subproblem optimality**: Assume the last multiplication taken place is multiplying $(A_0 \ldots A_i)$ by $(A_{i+1} \ldots A_{n-1})$.
  - Then the optimal solution $\boldsymbol{N_{0,n-1}}$ is the sum of two optimal subproblems, $\boldsymbol{N_{0,i} + N_{i+1,n-1}}$ plus the time for the last multiply.
  - If the global optimum did not have these optimal subproblems, we could define an even better "optimal" solution.

A Characterizing Equation



- Again assume the last multiplication is
  $(A_0 \ldots A_i)\,(A_{i+1} \ldots A_{n-1})$.
  - That is, we break at index i
- Consider all possible places for that final multiply (possible values of $0 \le i \le n-1$). That is…
  - $(A_0)(A_1 A_2 \ldots A_{n-1})$, and $(A_0\ A_1)(A_2 \ldots A_{n-1})$, and
  - $(A_0 A_1 A_2)(A_3 \ldots A_{n-1})$, $(A_0 \ldots A_3)(A_4 \ldots A_{n-1})$ etc till
  - $(A_0\ A_{n-2})(A_{n-1})$,
- Recall that A_i is a $d_i \times d_{i+1}$ dimensional matrix.
  - So, a characterizing equation for $N_{i,j}$ is the following:

$$N_{i,j} = \min_{i \le k < j}\{N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$$

I.e, break $(A_i \ldots A_j)$, into $(A_i \ldots A_k)(A_{k+1} \ldots A_j)$,

A Dynamic Programming Algorithm

Since subproblems overlap, we don't use recursion.

Instead, we construct optimal subproblems "bottom-up."

$N_{i,i}$'s are easy, so start with them

Then do length 2,3,… subproblems, and so on.

Running time:

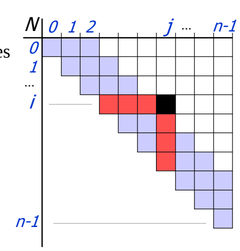
$O(n^3)$

```
Algorithm matrixChain(S):
Input:  sequence S of n matrices to be
        multiplied
Output: # of multiplications in optimal
        parenthesization of S
 for i ← 1 to n-1 do  N_{i,i} ← 0
 for b ← 1 to n-1 do        //length of a run
     for i ← 0 to n-b-1 do  //start of run
         j ← i+b            //end of run
         N_{i,j} ← +∞
         for k ← i to j-1 do  //break pnt
             N_{i,j} ←min{N_{i,j} ,
                 N_{i,k} +N_{k+1,j}+d_i d_{k+1} d_{j+1}}
```

A Dynamic Programming Algorithm Visualization

$$N_{i,j} = \min_{i \le k < j}\{N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$$

- The bottom-up construction fills in the $\boldsymbol{N}$ array by diagonals
- $N_{i,j}$ gets values from pervious entries in i-th row and j-th column
- Filling in each entry in the $\boldsymbol{N}$ table takes $O(n)$ time.
- Total run time: $O(n^3)$
- Getting actual parenthesization can be done by remembering "k" for each $\boldsymbol{N}$ entry (next slide).



Matrix Chain algorithm

How do we find the actual order of operations ?

```
Algorithm matrixChain(S):
  for i ← 0 to n-1 do
    N_{i,i} ← 0
  for b ← 1 to n-1 do  //length of a run
    for i ← 0 to n-b-1 do //start of run
      j ← i+b             //end of run
      N_{i,j} ← +infinity
      for k ← i to j-1 do
        sum = N_{i,k} +N_{k+1,j} +d_i d_{k+1} d_{j+1}
        if (sum < N_{i,j} ) then
          N_{i,j} ← sum
          O_{i,j} ← k
  return N_{0,n-1}
```

Example: ABCD

- A is 10 × 5
- B is 5 × 10
- C is 10 × 5
- D is 5 × 10

N	0	1	2	3
0	**0** A	**500** **0** AB	**500** **0** A(BC)	**1000** **2** (A(BC))D
1		**0** B	**250** **0** BC	**500** **1** (BC)D
2			**0** C	**500** **0** CD
3				**0** D

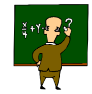
Recovering operations

- **Example: ABCD**
  - A is 10 × 5
  - B is 5 × 10
  - C is 10 × 5
  - D is 5 × 10

N	0	1	2	3
0	**0** A	**500** 0 AB	**500** 0 A(BC)	**1000** 2 (A(BC))D
1		**0** B	**250** 0 BC	**500** 1 (BC)D
2			**0** C	**500** 0 CD
3				**0** D

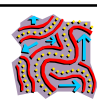
```
// return expression for multiplying
// matrix chain A_i through A_j

exp(i,j)
  if (i=j) then          // base case, 1 matrix
    return 'A_i'
  else
    k = O[i,j]           // see red values on left
    S1 = exp(i,k)        // 2 recursive calls
    S2 = exp(k+1,j)
    return '(' S1 S2 ')'
```

The General Dynamic Programming Technique

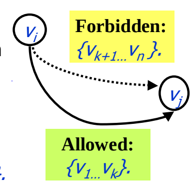


- Applies to a problem that at first seems to require a lot of time (often exponential), provided we have:
  - **Simple subproblems:** the subproblems can be defined in terms of a few variables, such as j, k, l, m, and so on.
  - **Subproblem optimality:** the global optimum value can be defined in terms of optimal subproblems

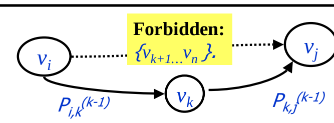
Example 3: All-Pairs Shortest Paths Floyd-Warshall alg



- Given a graph $G(V,E)$ with weights (positive and negative) assign to each edges. Assume $V=\{v_1 \ldots v_n\}$.
- Compute a matrix D such that $D[i,j]$ contains the length of the shortest path from v_i to v_j.
- Define $P_{i,j}^{(k)}$ as the shortst path $v_i \rightarrow v_j$ that does **not** go through any of the vertices $\{v_{k+1\ldots}v_n\}$. (that is, it is **allowed** to go through any of $\{v_{1\ldots}v_k\}$.
- $D_k[i,j]$ – the length of $P_{i,j}^{(k)}$
- We compute D_0 first, then D_1, etc.

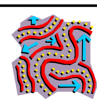


This example appears is in the shortest paths'' chapter of CLRS (25.2)

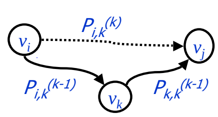


- Assume $D_{k-1}[i,j]$ has been computed (**$1 < i, j < n$**).
- We now want to compute $D_k[i,j]$. I.e. now we **can** (but don't have to) go through v_k on the shortest path $v_i \rightarrow v_j$.
- *Two possibilities:*
  - Going through v_k is longer, and better stick to $P_{i,j}^{(k-1)}$. (previous found shortest path $v_i \rightarrow v_j$)
  - Use $P_{i,k}^{(k-1)}$, the shortest path $v_i \rightarrow v_k$ to reach v_k , and continue along $P_{i,k}^{(k-1)}$ to v_j.
- $D_k[i,j] = \min(D_{k-1}[i,j],\ \ D_{k-1}[i,k] + D_{k-1}[k,j])$

Floyd Warshll-Pairs Shortest Paths Computing $D_k[i,j]$ for every i,j,k.

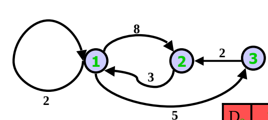


Algorithm *AllPair*(*G*) for all vertex pairs (i,j)
if $i = j$ then $D_0[i,i] \leftarrow 0$
else if (v_i, v_j) is an edge in G
  $D_0[i,j] \leftarrow w(v_i, v_j)$
else
  $D_0[i,j] \leftarrow +\infty$



for $k \leftarrow 1$ to n do
  for $i \leftarrow 1$ to n do
    for $j \leftarrow 1$ to n do
      $D_k[i,j] = \min\{\ D_{k-1}[i,j],\ D_{k-1}[i,k] + D_{k-1}[k,j]\ \}$
return D_n

Floyd's algorithm: example



D_0	1	2	3
1	0	8	5
2	3	0	∞
3	∞	2	0

D	1	2	3
1	0 0+0	8 0+8	5 0+5
2	3 3+0	0 3+8	8 3+5
3	∞ ∞+0	2 ∞+8	0 ∞+5

D	1	2	3
1	0 8+3	8 8+0	5 8+8
2	3 0+3	0 0+0	8 0+8
3	5 2+3	2 2+0	0 2+8

D	1	2	3
1	0 5+5	7 5+2	5 5+0
2	3 8+5	0 8+2	8 8+0
3	5 0+5	2 0+2	0 0+0

Example 4: Edit distance

Given strings x,y, the **edit distance** ***ed***(x,y) between x and y is defined as the minimum number of operations that we need to perform on x, in order to obtain y.

Defintion: An Operations (in this context)
Insertion/Deletion/Replacement of a **single** character.

Examples:
ed("aaba", "aaba") = 0
ed("aaa", "aaba") = 1
ed("aaaa", "abaa") = 1
ed("baaa", "") =4
ed("baaa", "aaab") =2

Example 4': ``Priced'' Edit distance *ed(x,y)*

Assume also given

- *InsCost*, - the cost of a single **insertion** into x.
- *DelCost* - the cost of a single **deletion** from x, and
- *RepCost* - the cost of **replacing** one character of x by a different character.

Definition: Given strings x,y, the **edit distance** ***ed***(x,y) between x and y is the cheapest sequence of operations, starting on x and ending at y.

Problem: Compute ***ed***(x,y), and compute the sequence of operations.

Theorem:

Let $c[i,j]$ = ed($x[1..i], y[1..j]$) then

If $x[i]=y[j]$ then $c[i,j] = c[i-1,j-1]$

If $X[i]\neq Y[j]$ then $c[i,j] = \min\{ c[i-1, j] + InsCost,\ c[i,j-1] + DelCost,\ c[i-1,j-1] + RepCost \}$

Algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

```
ed(x, y)
  for i=0 to m   c[i, 0] = 0
  for j=0 to n   c[0, j] = 0

  for i=1 to m
    for j=1 to n
      if (x[i] == y[j] )
        then c[i, j] ← c[ i–1, j–1]
        else c[i, j] ←min{    c[ i-1 , j ]   +     InsCost,
                              c[ i-1, j-1 ] +     RepCost,
                              c[ i , j-1]    +     DelCost
                            }
```

Time = $\Theta(m\ n)$ = constant work per table entry. Space = $\Theta(m\ n)$.