CS 545

Shortest Paths in Graphs

Alon Efrat

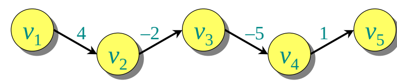
Slides courtesy of Erik Demaine with small by Carola Wenk and Alon Efrat

Paths in graphs

Consider a digraph $G = (V, E)$ with edge-weight function $w : E \to \mathbb{R}$. The ***weight*** of path $p = v_1 \to v_2 \to \cdots \to v_k$ is defined to be

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}).$$

Example:



$w(p) = -2$

Shortest paths

A ***shortest path*** from u to v is a path of minimum weight from u to v. The ***shortest-path weight*** from u to v is defined as

$\delta(u, v) = \min\{w(p) : p$ is a path from u to $v\}$.

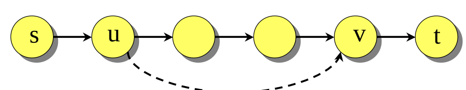
Also called ***distance*** of ***u*** from ***v***

Note: $\delta(u, v) = \infty$ if no path from u to v exists.

Optimal substructure

Theorem. A subpath of a shortest path is a shortest path.

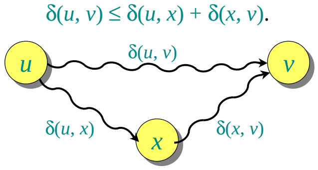
Proof. Cut and paste:



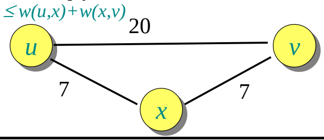
Triangle inequality

Theorem. For all $u, v, x \in V$, we have
$\delta(u, v) \le \delta(u, x) + \delta(x, v)$.

Proof.



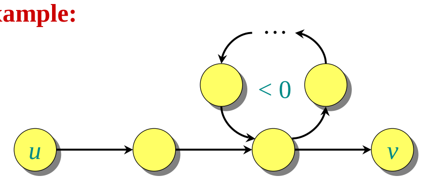
Note: This does not imply that
$w(u,v) \le w(u,x)+w(x,v)$



Well-definedness of shortest paths

If a graph G contains a negative-weight cycle, then some shortest paths may not exist.

Example:



Single-source shortest paths

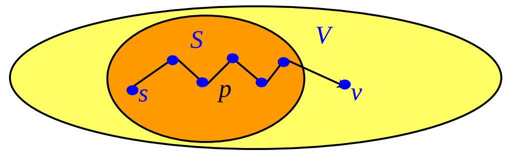
Problem. From a given source vertex $s \in V$, find the shortest-path weights $\delta(s, v)$ for all $v \in V$.

If all edge weights $w(u, v)$ are *nonnegative*, all shortest-path weights must exist. We'll use **Dijkstra**'s algorithm.

IDEA: Greedy.
1. Maintain a set S of vertices whose shortest-path distances from s are known. Also maintain **distance estimates** to the other vertices.
2. At each step add to S the vertex $v \in V - S$ whose distance estimate from s is minimal.
3. Update the distance estimates of vertices adjacent to v.

Internal paths - definition

1. Let S be a set of vertices (that contains s)
2. We say that path p is **internal** to S if all its vertices, excluding maybe the last one, are in S.
3. Distance estimation: The algorithm maintains for every vertex v the value $d[v]$, which is the length of the shortest path from s to v, which is internal to S.
4. Will show: If v is in S, then $d[v] = \delta(s, v)$



Dijkstra's algorithm

$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
  do $d[v] \leftarrow \infty$
$S \leftarrow \varnothing$
$Q \leftarrow V$ ⊳ Q is a priority queue maintaining $V - S$
while $Q \neq \varnothing$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ /* all nbrs of u */
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$ } *relaxation step*

Implicit DECREASE-KEY

Dijkstra's algorithm

$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
  do $d[v] \leftarrow \infty$
$S \leftarrow \varnothing$
$Q \leftarrow V$ ⊳ Q is
while $Q \neq \varnothing$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$ *relaxation step*

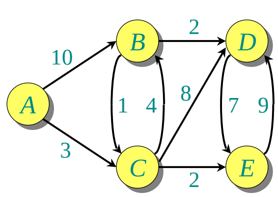
Implicit DECREASE-KEY

PRIM's algorithm

$Q \leftarrow V$
$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$
while $Q \neq \varnothing$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    for each $v \in Adj[u]$
      do if $v \in Q$ and $w(u, v) < key[v]$
        then $key[v] \leftarrow w(u, v)$
          $\pi[v] \leftarrow u$

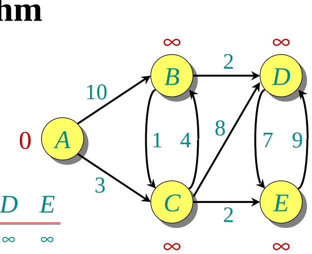
Example of Dijkstra's algorithm

Graph with nonnegative edge weights:



Example of Dijkstra's algorithm

Initialize:

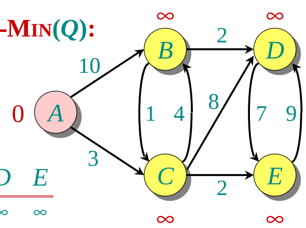


Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: { }

Example of Dijkstra's algorithm

"A" ← EXTRACT-MIN(Q):

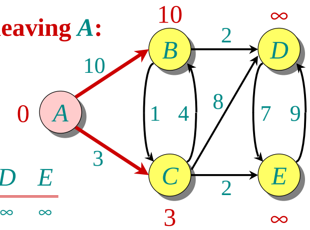


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞

S: { *A* }

Example of Dijkstra's algorithm

Relax all edges leaving *A*:

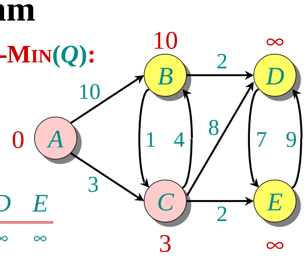


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	–	–

S: { *A* }

Example of Dijkstra's algorithm

"C" ← EXTRACT-MIN(Q):

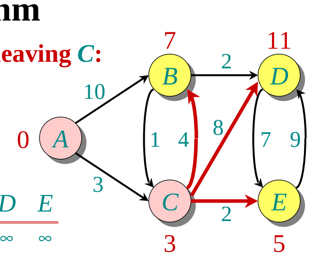


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	–	–

S: { *A*, *C* }

Example of Dijkstra's algorithm

Relax all edges leaving *C*:

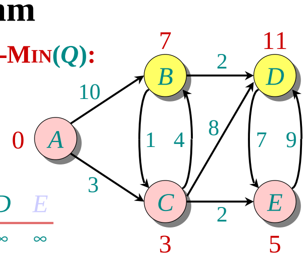


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	–	–
		7		11	5

S: { *A*, *C* }

Example of Dijkstra's algorithm

"E" ← EXTRACT-MIN(Q):

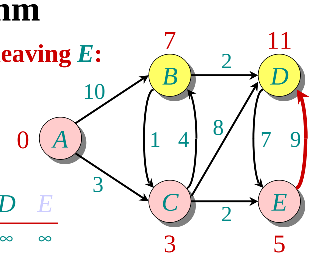


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	–	–
		7		11	5

S: { *A*, *C*, *E* }

Example of Dijkstra's algorithm

Relax all edges leaving *E*:

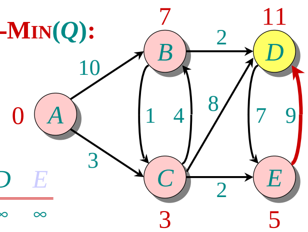


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { *A*, *C*, *E* }

Example of Dijkstra's algorithm

"B" ← EXTRACT-MIN(Q):

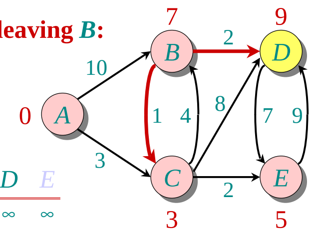


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E, B }

Example of Dijkstra's algorithm

Relax all edges leaving B:

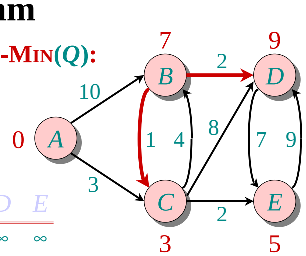


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { A, C, E, B }

Example of Dijkstra's algorithm

"D" ← EXTRACT-MIN(Q):



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { A, C, E, B, D }

Correctness — Part I

Lemma. At any stage of the algorithm, and for every vertex v, it is always true that $d[v] \geq \delta(s, v)$.

Proof. It is true after initialization (trivially).
Suppose not. Let v be the first (chronologically) vertex for which $d[v] < \delta(s, v)$, and let u be the vertex that caused $d[v]$ to change: $d[v] = d[u] + w(u, v)$. Then,

$d[v]$	$< \delta(s, v)$	supposition
	$\leq \delta(s, u) + \delta(u, v)$	triangle inequality
	$\leq \delta(s,u) + w(u, v)$	sh. path ≤ specific path
	$\leq d[u] + w(u, v)$	v is first violation

Contradiction. □

Handwave: $d[v]$ is the length of **a** path to v, while $\delta(s, v)$ is the **shortst** path to v.

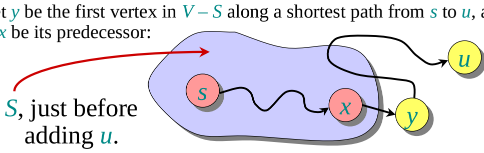
Correctness — Part II

Theorem. When the algorithm terminates, $d[v] = \delta(s, v)$, $\forall v \in V$.

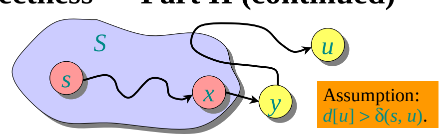
Proof. It suffices to show that $d[v] = \delta(s, v)$, when v is added to S.

- Suppose u is the first vertex added to S for which $d[u] > \delta(s, u)$.
  - Recall: $d[u] \geq \delta(s, u)$ always.
  - Recall $d[u] \leq d[w]$, $\forall w \in V\text{-}S$

- Let y be the first vertex in $V - S$ along a shortest path from s to u, and let x be its predecessor:



Correctness — Part II (continued)



- Since u is the first vertex violating the claimed invariant, $d[x] = \delta(s, x)$.
- Since subpaths of shortest paths are shortest paths, $\delta(s, y)= \delta(s, x) + w(x, y)$

- When x joined S, we perform a relaxation step: $d[y]=\min\{d[y], d[x]+w(x,y)\}$ so $d[y]=\delta(s, y)$
- If u is y we are done. So assume u is **not** y.

- We have $d[y] = \delta(s, y) \leq \delta(s, u) < d[u]$. But, $d[u] \leq d[y]$ by our choice of u, a contradiction. □

Analysis of Dijkstra

$|V|$ times:

while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    $degree(u)$ times:
    for each $v \in Adj[u]$
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$

Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

Time = $\Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

Note: Same formula as in the analysis of Prim's minimum spanning tree algorithm.

Analysis of Dijkstra (continued)

Time = $\Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
Fibonacci heap	$O(\lg V)$ amortized	$O(1)$ amortized	$O(E + V \lg V)$ worst case

Unweighted graphs

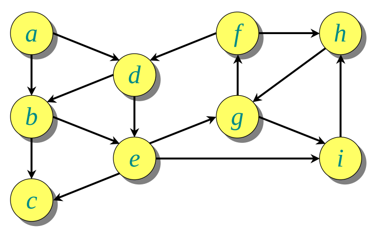
Suppose $w(u, v) = 1$ for all $(u, v) \in E$. Can the code for Dijkstra be improved?

- Use a simple FIFO queue instead of a priority queue.
- ***Breadth-first search***

while $Q \neq \emptyset$
  do $u \leftarrow$ DEQUEUE(Q)
    for each $v \in Adj[u]$
      do if $d[v] = \infty$
        then $d[v] \leftarrow d[u] + 1$
          ENQUEUE(Q, v)

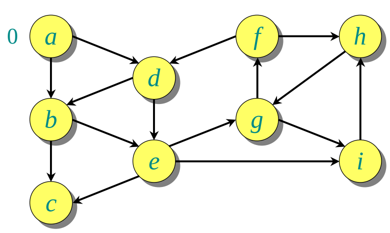
Analysis: Time = $O(V + E)$.

Example of breadth-first search



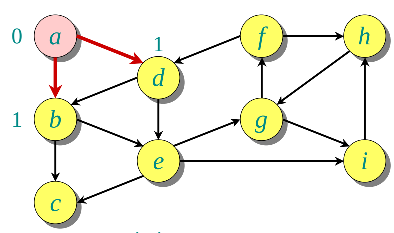
Q:

Example of breadth-first search



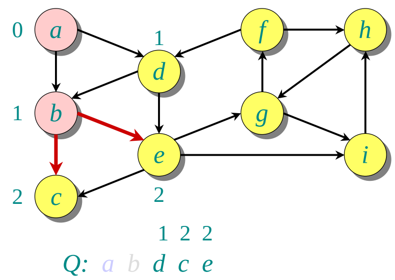
Q: a (0)

Example of breadth-first search

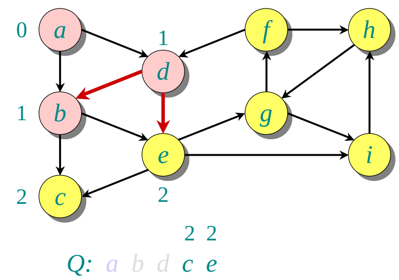


Q: a b (1) d (1)

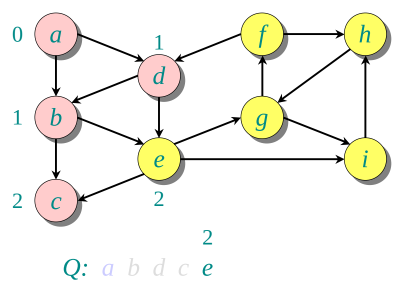
Example of breadth-first search



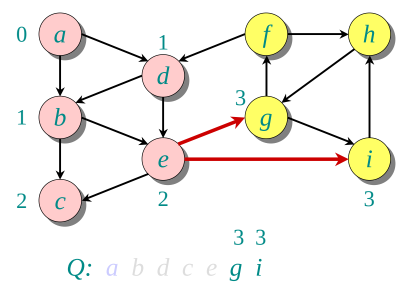
Example of breadth-first search



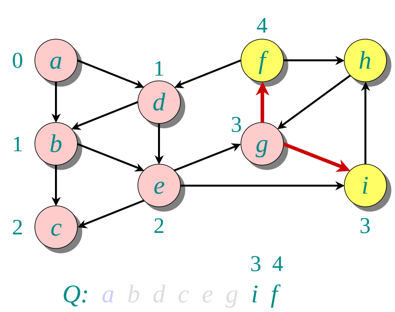
Example of breadth-first search



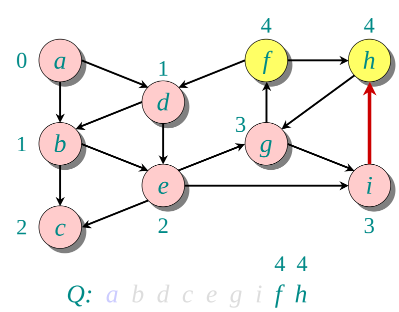
Example of breadth-first search



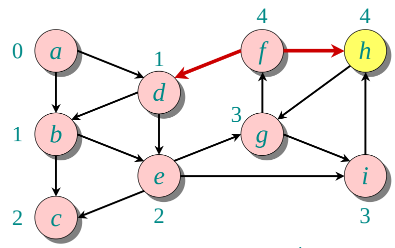
Example of breadth-first search



Example of breadth-first search



Example of breadth-first search



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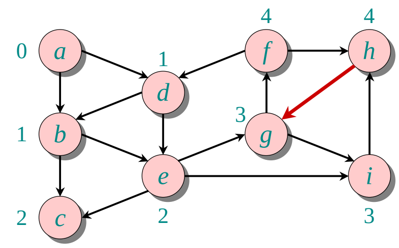
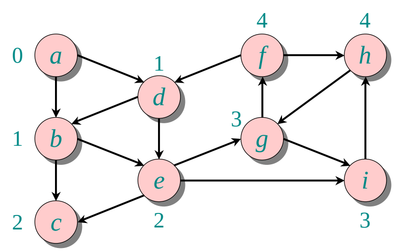
Q: *a b d c e g i f h*

Example of breadth-first search



Q: *a b d c e g i f h*

Example of breadth-first search



Q: *a b d c e g i f h*

Correctness of BFS

while $Q \neq \emptyset$
  do $u \leftarrow$ DEQUEUE(Q)
    for each $v \in Adj[u]$
      do if $d[v] = \infty$
        then $d[v] \leftarrow d[u] + 1$
          ENQUEUE(Q, v)

Key idea:

The FIFO Q in breadth-first search mimics the priority queue Q in Dijkstra.

- **Invariant:** v comes after u in Q implies that $d[v] = d[u]$ or $d[v] = d[u] + 1$.

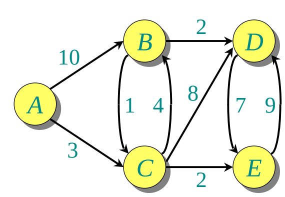
How to find the actual shortest paths?

Store a predecessor tree:

$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
  do $d[v] \leftarrow \infty$
$S \leftarrow \emptyset$
$Q \leftarrow V$ □ Q is a priority queue maintaining $V - S$
while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$
          $\pi[v] \leftarrow u$ /* *Producing edges of the shortest paths tree* */

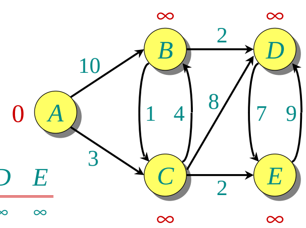
Example of Dijkstra's algorithm

Graph with nonnegative edge weights:



Example of Dijkstra's algorithm

Initialize:

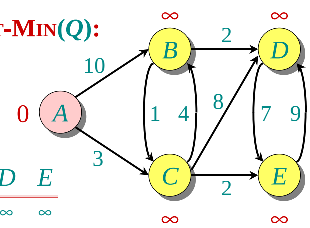


Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: {}

Example of Dijkstra's algorithm

"A" ← EXTRACT-MIN(Q):



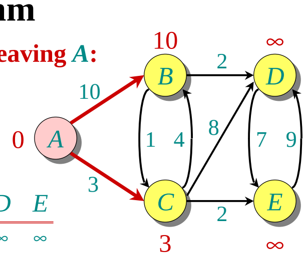
Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: { A }

π:	A	B	C	D	E
	–	–	–	–	–

Example of Dijkstra's algorithm

Relax all edges leaving A:



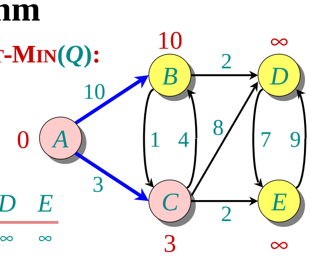
Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	–	–

S: { A }

π:	A	B	C	D	E
	–	A	A	–	-

Example of Dijkstra's algorithm

"C" ← EXTRACT-MIN(Q):



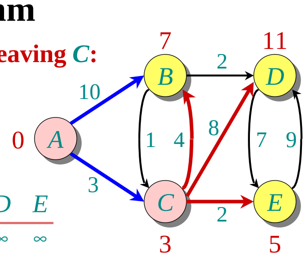
Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	–	–

S: { A, C }

π:	A	B	C	D	E
	–	A	A	–	-

Example of Dijkstra's algorithm

Relax all edges leaving C:



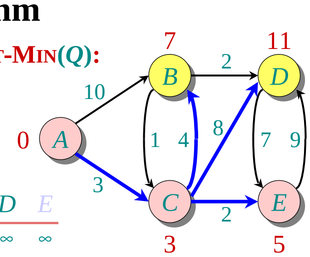
Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	–	–
		7		11	5

S: { A, C }

π:	A	B	C	D	E
	–	C	A	C	C

Example of Dijkstra's algorithm

"E" ← EXTRACT-MIN(Q):



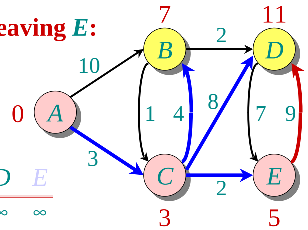
Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	–	–
		7		11	5

S: { A, C, E }

π:	A	B	C	D	E
	–	C	A	C	C

Example of Dijkstra’s algorithm

Relax all edges leaving *E*:



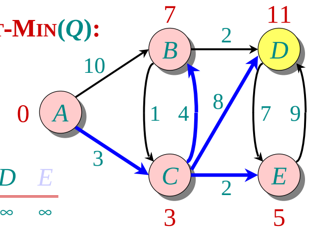
Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { *A*, *C*, *E* }

π:	*A*	*B*	*C*	*D*	*E*
	–	C	A	C	C

Example of Dijkstra’s algorithm

“*B*” ← EXTRACT-MIN(*Q*):



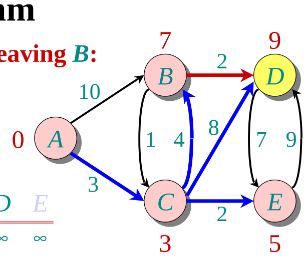
Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { *A*, *C*, *E*, *B* }

π:	*A*	*B*	*C*	*D*	*E*
	–	C	A	C	C

Example of Dijkstra’s algorithm

Relax all edges leaving *B*:



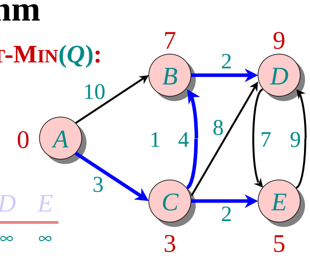
Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { *A*, *C*, *E*, *B* }

π:	*A*	*B*	*C*	*D*	*E*
	–	C	A	B	C

Example of Dijkstra’s algorithm

“*D*” ← EXTRACT-MIN(*Q*):



Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

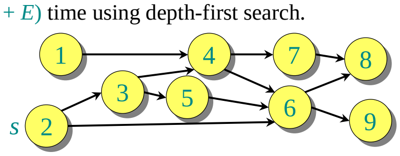
S: { *A*, *C*, *E*, *B*, *D* }

π:	*A*	*B*	*C*	*D*	*E*
	–	C	A	B	C

DAG shortest paths

If the graph is a ***directed acyclic graph*** (***DAG***), we first ***topologically sort*** the vertices:

- Determine $f : V \to \{1, 2, \ldots, |V|\}$ such that $(u, v) \in E$ $\Rightarrow f(u) < f(v)$ (will describe later how).
- $O(V + E)$ time using depth-first search.



Walk through the vertices $u \in V$ in this order, relaxing the edges in $Adj[u]$, thereby obtaining the shortest paths from s in a total of $O(V + E)$ time.