CS 545

Finding the closest pair of points

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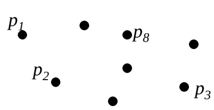
A Simple Randomized Sieve Algorithm for the Closest-Pair Problem

***Inf. Comput.* 118(1): 34-37 (1995)**

Problem definition

Given: A set $S=\{p_1,\ldots p_n\}$ of n points in the plane

Problem: Find the pair $p_i p_j$ that minimizes $d(p_i,p_j)$, where $d(p_i, p_j)$ is the Euclidean distance between p_i and p_j.



$O(n^2)$ time algorithm – trivial

$\Omega(n \log n)$ bound for any deterministic algorithm.

In this talk – a randomized algorithm whose expected running time is $O(n)$

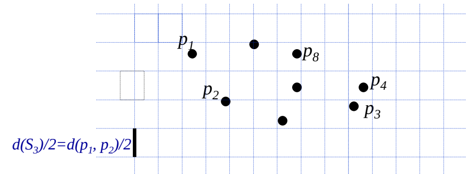
Notation

Let $S_i=\{p_1, p_2, \ldots p_i\}$

Let $d(S_i)$ denote the distance between the closest pair in S_i

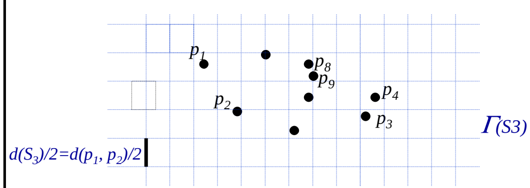
Clearly $d(S_2) \geq d(S_3) \geq d(S_4) \geq \ldots \geq d(S_n)$

Idea – incremental algorithm – compute $d(S_{i+1})$ from $d(S_i)$



Let $\Gamma(S_i)$ denote an axis-parallel grid, where the edge-length of each grid-cell is $d(S_i)/2$, and one of its corner is on the point $(0,0)$

Properties of $\Gamma(S_i)$.

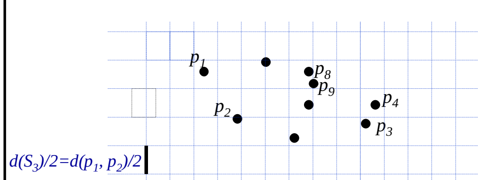


Claim 1: there is at most one point of S_i inside every cell of $\Gamma(S_i)$.

Proof – if there are two, then the distance between them is smaller than the length of the diagonal of the cell, which is

$$(\sqrt{2})d(S_i)/2 = d(S_i)/\sqrt{2} < d(S_i)$$

Locating points.

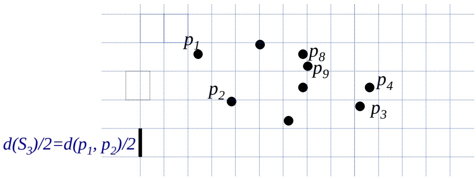


Claim 2: given $d(S_i)$ we can place all points of S_i in a data structure $H(S_i)$, such that we can (in $O(1)$ expected time)

1) insert a new point p_j
2) Given a query point q find if there is a point of S_i in the cell of $\Gamma(S_i)$ containing q.

The structure $H(S_i)$ is described in HW1.

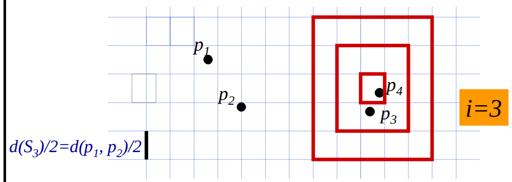
Rehashing with $d(S_i)$



Procedure **Rehashing with $d(S_i)$** :
Construct the hash table $H(S_i)$ (from HW1) with $d(S_i)$, and inserting all points of S_i into the table.

Expected time O(| S_i|).

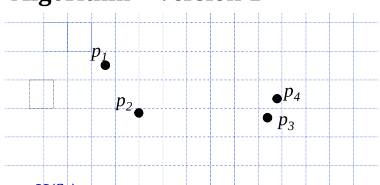
Deciding if $d(S_i) > d(S_{i+1})$



To decide whether $d(S_i)>d(S_{i+1})$ or $d(S_i)=d(S_{i+1})$ do
find all points of S_i in the cell containing $p_{i+1\ldots}$
and in all the cells whose distance from this cell < $d(S_i)$
Measure the distance from p_{i+1} to each of these points.

Note – only a constant number of cells, and due to Claim 1, only a constant number of points. Altogether: (expected) constant time .

Algorithm – version 1



Input: S
Output: $d(S)$, The closest pair of S

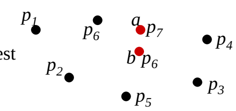
Find $d(S_2)$, and construct $H(S_2)$.
For $i=2,4\ldots n-1$ do {
Use $H(S_i)$ to decide whether $d(S_i)>d(S_{i+1})$ or $d(S_i)=d(S_{i+1})$

If $d(S_i)=d(S_{i+1})$ then $\Gamma(S_{i+1}) = \Gamma(S)$; insert p_{i+1} ; and
$H(S_{i+1}) = H(S_i)$. No work is needed. (constant time)
Else /*$d(S_i)>d(S_{i+1})$*/ rehash with $d(S_{i+1})$. ($O(i)$ expected time)
}
Running time: Worst case $1+2+3+\ldots(n-1) = O(n^2)$

Algorithm – version 2

Create random permutation of the points of S before calling the algorithm of version 1.

Assumption: the closest pair is unique.



Claim 3: The probability that $d(S_i)>d(S_{i+1})$ is $2/(i+1)$.
Proof: There are $(i+1)$ points, two are special (determining the closest pair. All permutations are equally likely, so the probability that one of the special pair appears last in the permutation is $2/(i+1)$.

Finishing the analysis

So in the ***i'***th stage we are spending $O(1)$ time with probability $(i-1)/(i+1)$, and $O(i)$ time with probability $2/(i+1)$, so the expected work in this stage is $O(i)\ \ 2/(i+1) = O(1)$.

Hence the total expected time is $O(n)$.

This argument can be done more formal using random variables.

Expected time

Let T_i denote the expected time at stage i. Then $T_i=1$ with probability $(i-1)/(i+1)$ and $T_i=i$ with probability $2/(i+1)$

$$E(T_i)=\sum_{i=1}^{\infty} i\Pr(T_i=i)=1\Pr(T_i=i)+i\Pr(T_i=i)=2$$

Hence the total time is

$$E(\sum_{i=3}(T_i))=\sum_{i=3}^{n}E(T_i=i)=\sum_{i=3}^{n}2=O(n)$$