Fibonacci Heap

Thanks to Sartaj Sahni for the original version of the slides

Priority Queues

		Heaps			
Operation	Linked List	Binary	Binomial	Fibonacci *	Relaxed
make-heap	1	1	1	1	1
insert	1	log N	log N	1	1
find-min	N	1	log N	1	1
delete-min	N	log N	log N	log N	log N
union	1	N	log N	1	1
decrease-key	1	log N	log N	1	1
delete	N	log N	log N	log N	log N
is-empty	1	1	1	1	1

Dijkstra/Prim
1 make-heap
|V| insert
|V| delete-min
|E| decrease-key

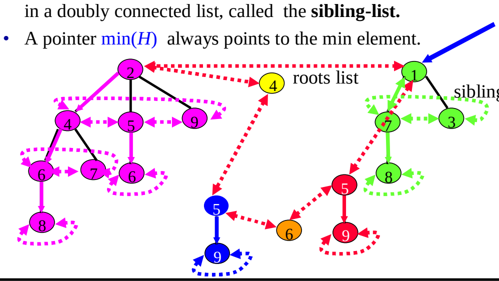
$O(|V|^2)$ — $O(|E| \log |V|)$ — $O(|E| + |V| \log |V|)$

Fibonacci heaps

- Similar to binomial heaps, consists of a collection of trees, each arranged in a heap-order (each node is smaller than each of its children)
- Unlike binomial heaps, can have many trees of the same cardinality, and a tree does not have to have exactly 2^i nodes.
- Main idea – **laziness** is welcomed. Try to postpone doing the hard work, until no other solution works.

General Structure

- Very similar to Bionomail heaps
- Main structure: A collection of trees, each in a heap-order.
- All root are stored in a doubly connected list, called the **roots-list.**
- Every node points to one of its childrens. All the children are stored in a doubly connected list, called the **sibling-list.**
- A pointer min(H) always points to the min element.



Node Structure

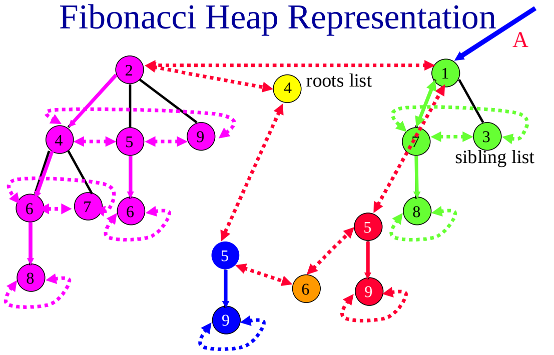
- Very similar to Bionomail heaps
- Each node v stores its
  - degree,
  - a points to its parent,
  - a points to a child,
  - data,
  - Pointers to left and right sibling used for circular doubly linked list of siblings, called the **sibling list**.

More – in next slide

Node Structure

- Each node v stores a flag **ChildCut** (not existing in binomial heaps)
  - **True** only if
    - v is not a root, and
    - v has lost a single child since became a child of its current parent.
  - We say that v is **marked** in this case.
- Will see: *Extract_Min* is the only operation that makes one node a child of another, and then flag might change.
  - Flag is undefined for a root node (not used)

Fibonacci Heap Representation



Potential Function

Some nodes would be marked (to be explained later)
We use the potential functions for the heap H
$\Phi(H) = t(H) + 2\ m(H)$

Where $t(H)$ is the number of trees in H
And $m(H)$ is the number of marked nodes in H.

Insert(x)

Create a new tree consisting of a single node v whose key is x,
Add v to the roots list. (always unmarked)

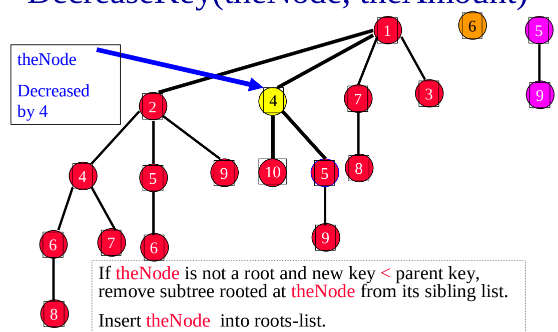
Actual time w_i needed for the operation is 1

Number of trees increased by 1.

Changes in potential function
$\Phi(H') - \Phi(H) = \Delta\Phi(H) = t(H') + 2\ m(H') - t(H) - 2\ m(H) = 1$

So the amortized work $a_i = w_i + \Delta\Phi(H) = 2$

DecreaseKey(theNode, theAmount)



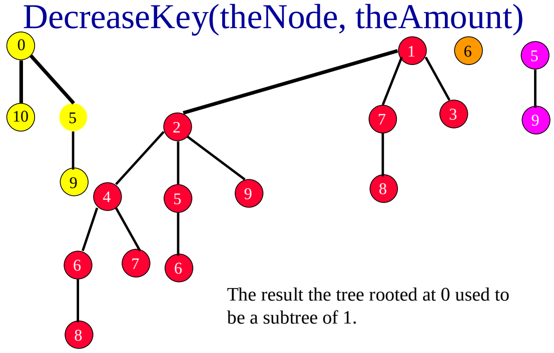
If theNode is not a root and new key < parent key, remove subtree rooted at theNode from its sibling list.

Insert theNode into roots-list.

Perform cascading_cut from parent(theNode)

(described later)

DecreaseKey(theNode, theAmount)



The result the tree rooted at 0 used to be a subtree of 1.

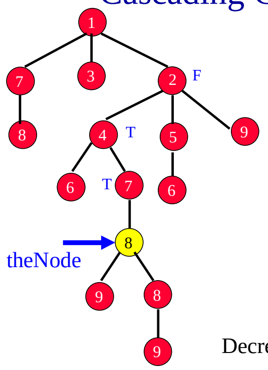
Convince yourself that all pointers modifications are doable in O(1)

Cascading Cut

- When theNode is cut out of its sibling list in a decrease key operation, follow path from parent of theNode upward toward the root.
- Encountered nodes (other than root) with ChildCut = true are cut from their sibling lists and inserted into roots-list.
- Stop at first node with ChildCut = false.
- For this node, set ChildCut = true. (since it just lost exactly one child)

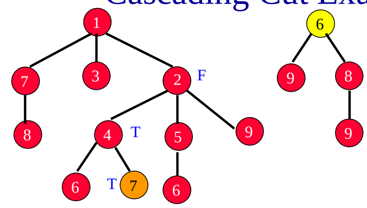
- *In other words, if a node lost two children since it became a child, it must move itself from the the parent to the roots-list.*

Cascading Cut Example

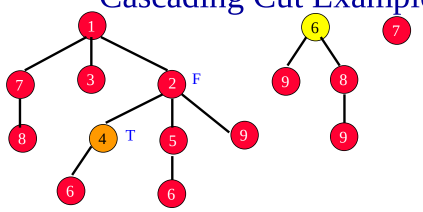


Decrease key by 2.

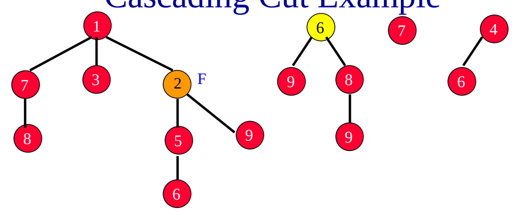
Cascading Cut Example



Cascading Cut Example

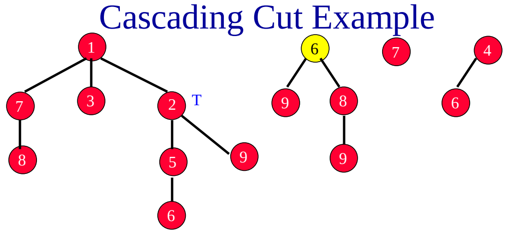


Cascading Cut Example



Note – a node that moves to the root lists looses its mark (becomes unmarked).

Cascading Cut Example



Actual time complexity of the cascading_cut of a path of length k is $\Theta(k+1)$ (can be $\Theta(h)$ in the worst case, where h is the height)

Assume we specify the time of an elementary operations, so that this time w_i is $k+1$.

Note that the number of trees increases by $k+1$, and the number of marked nodes decreases by either k or $k+1$

Amortized time

Note that the number of marked nodes decreases by k or $k+1$, and the number of trees increased by $k+1$.

Let H' to be H denote the heap before and after the Decrease_min operation, then $t(H') = t(H)+k+1$ and $m(H')=m(H)-k$

The change in the potential function is (denoting H' to be H after the Decrease_min) is

$\Phi(H') - \Phi(H) = (t(H') + 2m(H') – (t(H) + 2 m(H)) = -k+2$

And

$a_i = w_i + \Phi(H') - \Phi(H) = k+(-k+2)=2$

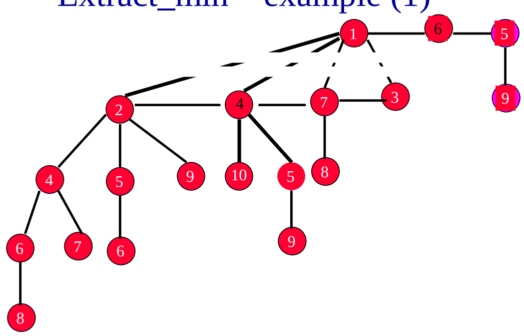
Deletion of a node *v*

- **Perform DecreaseKey(*v*) to value -∞**
- **Perform ExtractMin(*H*) – seen next.**

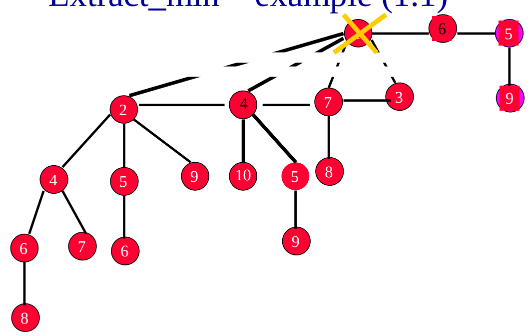
Extract_min

- Remember - there is a pointer (min*[H]*) pointing to the min.
- Set theNode= min*[H]*
- Moved all children of theNode to the the roots-list.
  - This is done by merging their sibling list with the root list (constant time)
- Remove theNode from its sibling list.
- Free theNode.
- Perform **Consolidate**(*H*) - merging trees.
  - /* This is a good time to reduce the number of trees */

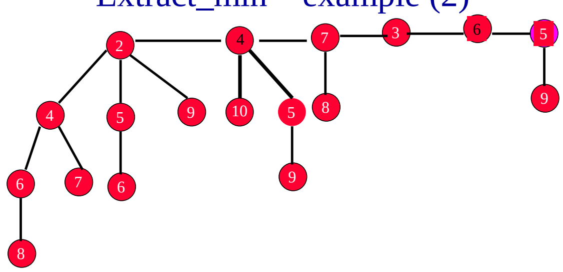
Extract_min – example (1)



Extract_min – example (1.1)



Extract_min – example (2)



Next comes the **consolidation**

Time analysis for Extract_min operation (not including consolidation (counted later))

- Let deg(*v*) is the number of children of *v*.
- Lemma: (CLRS 20.3): The number of nodes in a tree the tree rooted at *v* is $\geq \varnothing^{\deg(v)} \geq 1.5^{\deg(v)}$
- Conclusion 1: deg(*v*) =O(log *n*), for every node *v*.
- The actual time w_i needed for disconnecting *v* from its children and adding them to the roots list – O(log n)=deg(v)

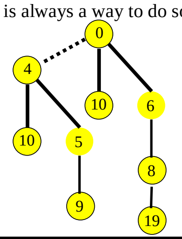
Union of two trees.
(Need for Consolidation)

•(Similar (but not identical) operation was seen in the binomial heaps)

•Degree of a tree is defined as the degree of the root of the tree.

•Given two trees with the same **degree of their roots,** connect the root of one as a child of the other root.

•There is always a way to do so while maintaining the heap order:



The root with larger key becomes the child of the smaller root

Point of potential confusion: For Binomial heaps, trees have the same size iff they have the same degree. Not true here

Extract_min – cont: Consolidation.

•Each extract_min is followed by the **consolidation** operation:

•This operation repeatedly joins trees with same degree, using the tree-union operation:

- •Repeatedly pick two trees with the same degree, and merge them:
- •…but trees are not sorted by degree, (as oppose to Binomial heaps) and there are many of them – how can this be done efficiently ? (on board)

•Finish when no two trees with the same degree exist.

•Recover the new minimum while doing so.

•Actual Time w_i – proportional to the number of trees

- •(since every operation reduces the number of tress by one, and takes a constant time) .

Time analysis for consolidation

- The consolidation takes actual time $t(H)$ time.
- In H', there at most one tree for each degree of its root (followed from conclusion 1), so $t(H')$ =O(log n).
- The number of marked nodes is not changed.
- $\Phi(H') - \Phi(H) = (t(H') + 2m(H')) - (t(H) + 2\, m(H)) = t(H')-t(H) = O(\log n)- t(H)$
- The amortized work is therefore

  $t(H)+ (\mathrm{O}(\log n) -t(H)) = \mathrm{O}(\log n)$

Time analysis for Delete

- Deletion consists of
  - first DecreaseKey (amortized time O(1)) and then
  - ExtractMin (amortized time O(log n))
- Total amortized work: $O(\log n)$

Toward proving lemma CLRS 20.3

- Claim: Let $F_0 = F_1=1$ and $F_{k+2}=F_{k+1}+F_k$. Then (induction) $F_{k+2} \geq \emptyset^{k}$
- Lemma 20.2: $F_{k+2} = 1 + \sum_{i=0}^{k} F_i$
- Proof – by induction, on the board.

- Lemma 20.1: Let x be a root, and let $y_1, \ldots, y_k$ denote its children, in the order they joined x. Then $\deg[y_i] \geq i-1$. (i=2,3…k).
- Proof:
  - When y_1 joined x, its degree was exactly i.
  - Since then its degree might have dropped to $i-1$ (this is where we needed the rule that an internal node might loose at most one child)

Proving lemma CLRS 20.3

- Let s_k denote the minimum number of nodes at a tree of degree k.
- **Lemma** 20.3 **:** $s_k \geq \emptyset^{k}$
- Proof: Let $y_1, \ldots, y_k$ denote its children of a node x, in the order they joined x.
  - The degree of y_i is $\geq i-1$,
  - hence containing $\geq F_{i-1}$ nodes,
  - $s_k = 1$ (root) + sum of number of nodes in subtrees