CS545
Introduction to Algorithms

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Some slides are courtesy of
Piotr Indyk and Carola Wenk

Polices

- From course webpage

Why study algorithms and performance?

- Performance often draws the line between what is feasible and what is impossible.
- Algorithmic mathematics provides a ***language*** for talking about program behavior.
  - (e.g., by using big-O –notation)
- In real life, many algorithms, though different from each other, fall into one of several ***paradigms*** (discussed shortly).
- These paradigms can be studied, and applied for new problems

Why these **particular** algorithms ??

- In this course, we will discuss problems, and algorithms for solving these problems.
- There are so many algorithms – why focus on the ones in the syllabus ?

Why these algorithms (cont.)

1. **Main paradigms:**
   a) **Greedy algorithms**
   b) **Divide-and-Conquers**
   c) **Dynamic programming**
   d) **Brach-and-Bound (mostly in AI)**
   e) **Etc etc.**
2. **Other reasons:**
   a) **Relevance to many areas:**
      - **E.g., networking, internet, search engines…**
   b) **Coolness**

The problem of sorting

Input: sequence $\langle a_1, a_2, \ldots, a_n\rangle$ of numbers.

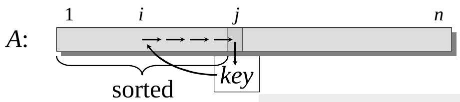
Output: permutation $\langle a'_1, a'_2, \ldots, a'_n\rangle$ such that $a'_1 \le a'_2 \le \cdots \le a'_n$.

Example:

Input: 8 2 4 9 3 6

Output: 2 3 4 6 8 9

Insertion sort



Invariants: *A[1..j-1]* is sorted

1 5 7 10 12 18 **9** *100 200*

1 5 7 10 12 18 100 200

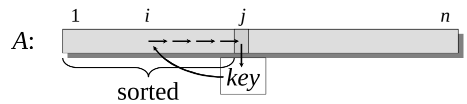
Consider *A[j]=9*. Not in the correct place.
Need to make room for 9.
We shift all elements right, starting from 10.

1 5 7 **9** *10 12 18 100 200*

Insertion sort

"pseudocode"

INSERTION-SORT (A, n) //input: $A[1 \,.\,.\, n]$
for $j \leftarrow 2$ **to** n *//outer loop*
do $key \leftarrow A[j]$
$i \leftarrow j - 1$
while $i > 0$ and $A[i] > key$ *//inner loop*
do { $A[i+1] \leftarrow A[i]$
$i \leftarrow i - 1$}
$A[i+1] = key$



Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

8 2 4 9 3 6
2 8 4 9 3 6

Example of insertion sort

8 2 4 9 3 6
2 8 4 9 3 6

Example of insertion sort

8 2 4 9 3 6
2 8 4 9 3 6
2 4 8 9 3 6

L1.13

Example of insertion sort

8 2 4 9 3 6
2 8 4 9 3 6
2 4 8 9 3 6

L1.14

Example of insertion sort

8 2 4 9 3 6
2 8 4 9 3 6
2 4 8 9 3 6
2 4 8 9 3 6

L1.15

Example of insertion sort

8 2 4 9 3 6
2 8 4 9 3 6
2 4 8 9 3 6
2 4 8 9 3 6

L1.16

Example of insertion sort

8 2 4 9 3 6
2 8 4 9 3 6
2 4 8 9 3 6
2 4 8 9 3 6
2 3 4 8 9 6

L1.17

Example of insertion sort

8 2 4 9 3 6
2 8 4 9 3 6
2 4 8 9 3 6
2 4 8 9 3 6
2 3 4 8 9 6

L1.18

Example of insertion sort

8	2	4	9	3	6	
2	8	4	9	3	6	
2	4	8	9	3	6	
2	4	8	9	3	6	
2	3	4	8	9	6	
2	3	4	6	8	9	*done*

L1.19

Running time

- The running time depends on the input: an already sorted sequence is easier to sort.
  - Parameterize the running time by the size of the input *n*
  - Seek upper bounds on the running time *T(n)* for the input size *n*, because everybody likes a guarantee.

L1.20

Kinds of analyses

Worst-case: (usually)
- $T(n)$ = maximum time of algorithm on any input of size n.

Average-case: (sometimes)
- $T(n)$ = expected time of algorithm over all inputs of size n.
- Need assumption of statistical distribution of inputs.

Best-case: (bogus)
- Cheat with a slow algorithm that works fast on *some* input.

L1.21

Machine-independent time

What is insertion sort's worst-case time?
- It depends on the speed of our computer:
  - relative speed (on the same machine),
  - absolute speed (on different machines).

BIG IDEA:
- Ignore machine-dependent constants.
- Look at ***growth*** of $T(n)$ as $n \to \infty$.

"Asymptotic Analysis"

L1.22

O-notation

Math:

we say that $T(n) = O(g(n))$ **iff**
there exists positive constants c_1, and n_0 such that
$0 \leq T(n) \leq c_1 g(n)$ for all $n \geq n_0$

Usually *T(n)* is running time, and *n* is size of input

Engineering:
- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 - 5n + 6046 = O(n^3)$

L1.23

Ω-notation

Math:

we say that $T(n) = \Omega(g(n))$ **iff**
there exists positive constants c_2, and n_0 such that
$0 \leq c_1 g(n) \leq T(n)$ for all $n \geq n_0$

Engineering:
- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 - 5n + 6046 = \Omega(n^3)$

L1.24

O-notation - cont

So if $T(n)= O(n^2)$ then we are also sure that
$T(n)= O(n^3)$ and that
$T(n)= O(n^{3.5})$ and
$T(n)= O(2^n)$

But it might or might not be true that $T(n)= O(n^{1.5})$.

However, if $T(n)= \Omega(n^2)$ then it is **not** true that
$T(n)= O(n^{1.5})$

Proofs: In the Homework

L1.25

Θ-notation

Math:
we say that $T(n)= \Theta(g(n))$ iff
there exist positive constants c_1, c_2, and n_0 such that
$0 \le c_1\, g(n) \le T(n) \le c_2\, g(n)$ for all $n \ge n_0$

In other words
$T(n)= \Theta(g(n))$ iff $T(n)= O(g(n))$ and $T(n)= \Omega(g(n))$

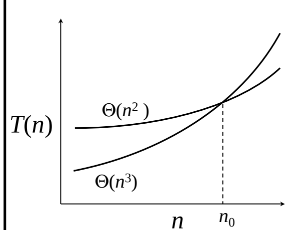
Engineering:

- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 – 5n + 6046 = \Theta(n^3)$

L1.26

Asymptotic performance

When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm.



- We shouldn't ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing of engineering objectives.
- Asymptotic analysis is a useful tool to help to structure our thinking.

L1.27

Insertion sort analysis

Worst case: Input reverse sorted.

T(n)=2c+3c+4c+…+c(n-1) = cn(n-1)/2

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2)$$ [arithmetic series]

Is insertion sort a fast sorting algorithm?
- Moderately so, for small n.
- Not at all, for large n.

L1.28

Merge sort (divide-and-conquer algorithm)

MERGE-SORT $A[1 \,.\,.\, n]$
1. If $n = 1$, done.
2. Recursively sort $A[\,1 \,.\,.\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,.\,.\, n\,]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: MERGE

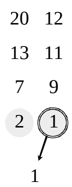
L1.29

Merging two sorted arrays

20	12
13	11
7	9
2	1

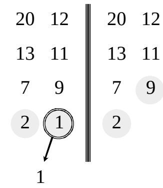
L1.30

Merging two sorted arrays



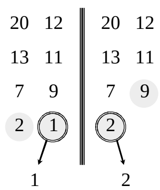
L1.31

Merging two sorted arrays



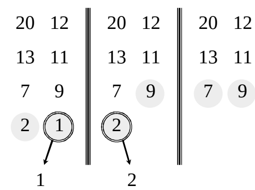
L1.32

Merging two sorted arrays



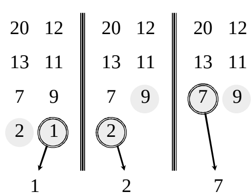
L1.33

Merging two sorted arrays



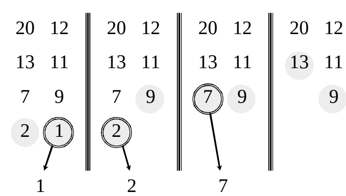
L1.34

Merging two sorted arrays



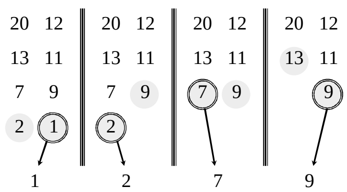
L1.35

Merging two sorted arrays



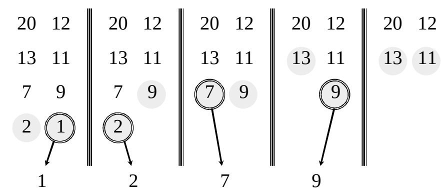
L1.36

Merging two sorted arrays



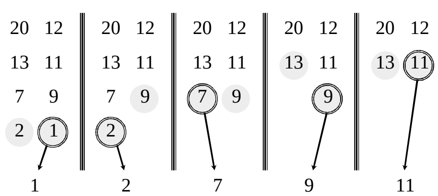
L1.37

Merging two sorted arrays



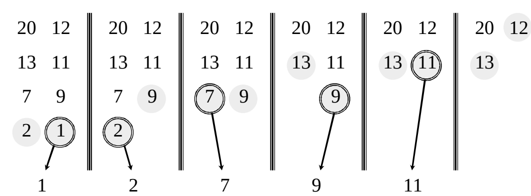
L1.38

Merging two sorted arrays



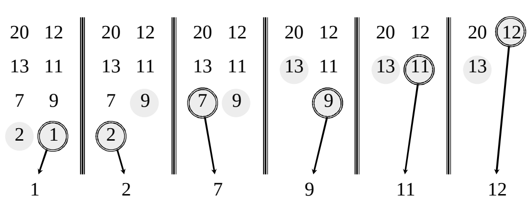
L1.39

Merging two sorted arrays



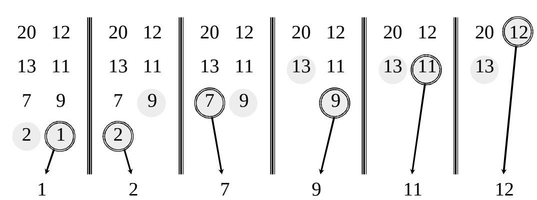
L1.40

Merging two sorted arrays



L1.41

Merging two sorted arrays



Time = $\Theta(n)$ to merge a total of n elements (linear time).

L1.42

Analyzing merge sort

$T(n)$	MERGE-SORT $A[1\,.\,.\,n]$
$\Theta(1)$	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1\,.\,.\lceil n/2\rceil\,]$ and $A[\,\lceil n/2\rceil+1\,.\,.\,n\,]$.
$\Theta(n)$	***3. "Merge"*** the 2 sorted lists

Abuse

Sloppiness: Should be $T(\lceil n/2\rceil) + T(\lfloor n/2\rfloor)$, but it turns out not to matter asymptotically.

L1.43

Recurrence for merge sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.
- CLRS provides several ways to find a good bound on $T(n)$.

L1.44

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

L1.45

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$T(n)$

L1.46

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$\boxed{cn}$

$\boxed{T(n/2)}$ $\boxed{T(n/2)}$

L1.47

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$\boxed{cn}$

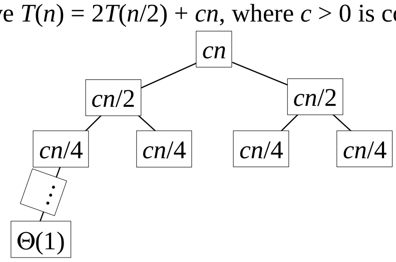
$\boxed{cn/2}$ $\boxed{cn/2}$

$\boxed{T(n/4)}$ $\boxed{T(n/4)}$ $\boxed{T(n/4)}$ $\boxed{T(n/4)}$

L1.48

Recursion tree

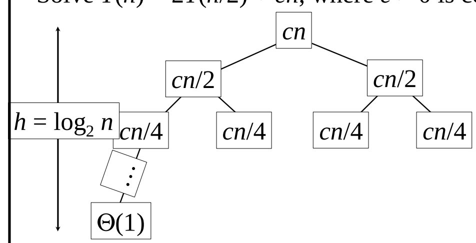
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



L1.49

Recursion tree

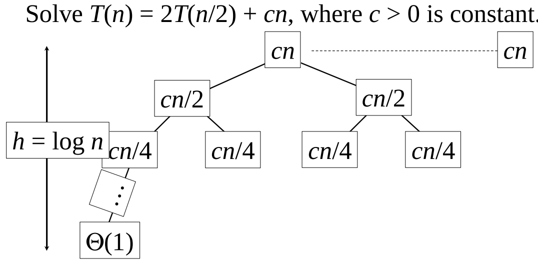
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



L1.50

Recursion tree

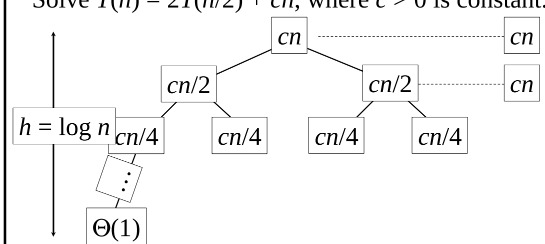
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



L1.51

Recursion tree

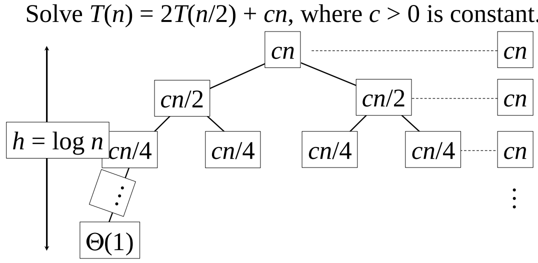
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



L1.52

Recursion tree

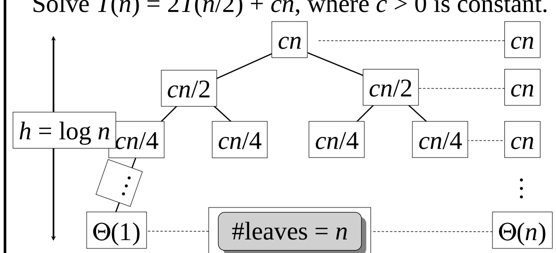
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



L1.53

Recursion tree

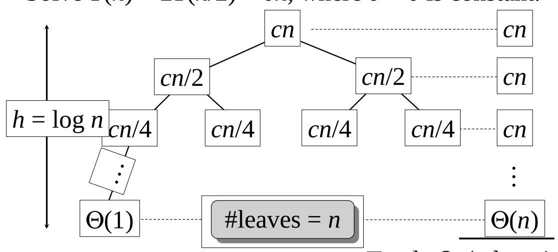
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



L1.54

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



Total $= \Theta(n \log n)$

Conclusions

- $\Theta(n \log n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so.
- Go test it out for yourself!

More examples – iterative method (1)

```
NoNeed(n){
    •   If (n<1) return ;
    •   Print(*)
    •   NoNeed(n-1)
}
```

Recursion formula: $T(n)=c+T(n-1)$, where $T(1)=c$. We can solve it using the **iteration method:**

$T(n)= c+T(n-1)=$
$c+\{c+T(n-2)\} = 2c+T(n-2) =$
$2c+\{ c+T(n-3) \} = 3c+T(n-3) =\ldots =$ (pick $k<n$)
$kc+T(n-k) =$ (setting $k = n-1$) …
$(n-1)c+T(1)=nc$

More examples (2)

```
1)   NoNeed(n){
     I.   if (n<1) return ;
     II.  for( i=1 ; i<n ; i++)   print(*)
     III. NoNeed(n-1)
2)   }
```

Recursion formula: $T(n)=cn+T(n-1)$, where $T(1)=c$. We can solve it using the **iteration method:**

$T(n)= cn+T(n-1)=$
$cn+\{c(n-1)+T(n-2)\} =$
$c[n+(n-1)]+\{c(n-2)+T(n-3)\}$
$=c[n+n-1+n-2+n-3]+T(n-3) =\ldots =$ (pick $k<n$)
$=c[n+n-1+n-2+ n-3+\ldots+n-k]+T(n-k-1) =$
(setting $k = n-1$) …
$c[n+ n-1+n-2 + n-3+\ldots+1]+T(1)=$
$c[1+2+3+\ldots +n]+T(1)= cn(n+1)/2 = \Theta(n^2)$

More examples (3)

```
I.   Read(n);   k=1 ;
II.  while( k ≤ n )   k=2k ;
```

- We know that each iteration takes $O(1)$ times. Need to find the number of iterations.
  - After the first iteration $k=2=2^1$
  - After the 2nd iteration $k=4=2^2$
  - After the 3rd iteration $k=8=2^3$
  - ….
  - After the j'th iteration $k=2^j$

Recall: $\log(ab)=\log(a)+\log(b)$
$\log(a^b) = b \log a$
$\log_a (x) = \log_b (x) / \log_b a$

- Assume j iterations occurs until the loop exits. After the last one we have that $k=2^j <2n$.
- Taking $\log_2$ from both sides, we have that
  $\log_2 k = \log_2(2^j) < \log_2(2n)$ or..
  $j \log_2 2< \log_2(2) + \log_2(n)$ or..
  $j< \log_2 n +1$ or $j=O (\log_2 n)$. $T(n)=O(\log n)$
- Homework: Prove $T(n)= \Theta(\log n)$

More examples (a bit tricky)

```
read(n)
for(i=1 ; i < n ;  i++)
  for( j=i ; j <n ; j += i )
      print( “*” ) ;
```

- Naïve analysis:
  - The outer loop (on i) runs exactly $n-1$ times
  - The inner loop (on j) runs $O(n)$ times.
  - Together $O(n^2)$ times.
- More “sensitive” analysis:
  - For $i=1$ we run through $j=1,2,3,4...n$, total n times.
  - For $i=2$ we run through $j=2,4,6,8,10\ldots n$, total $n/2$ times.
  - For $i=3$ we run through $j=3,6,9,12\ldots n$, total $n/3$ times .
  - For $i=4$ we run through $j=4,8,12,16\ldots n$, total $n/4$ times.
  - For $i=n$ we run through $j=n$, total $n/n=1$ time.
- Summing up: $T(n)=n+n/2+n/3+n/4+\ldots n/n =$
  $n(1+1/2+1/3+1/4+...1/n) \approx n \ln n$
  Harmonic sum

More examples: Geometric sum

```
read(n) ; a=0.31415926
while( n>1) {
    For( j=1; j<n ; j++ )  print("*")
    n=a*n ; }
```

•The **first** time the outer loop is called, the "print" is called n times.
•The **2nd** time the outer loop is called, the "print" is called an times.
•The **3rd** time the outer loop is called, the "print" is called a^2n times…
•The **k'th** time the outer loop is called, the "print" is called $a^k n$ times

•Let t be the number of iterations of the outer loop. Then the total time

$$= n + an + a^2n + a^3n + \ldots a^t n = n(1 + a + a^2 + a^3 + \ldots a^t) <$$
$$n(1 + a + a^2 + a^3 + \ldots a^t + \ldots) = n / (1-a) = O(n).$$

•Same analysis holds for any $a<1$

Recall: $1+a+a^2+\ldots+a^t = (1-a^{t+1})/(1-a)$.
If $a<1$ **then** $1+a+a^2+\ldots+a^t+\ldots = 1/(1-a)$

L1.61

Properties of big-O

- **Claim:** if $T_1(n)=O(g_1(n))$ and $T_2(n)=O(g_2(n))$ then $T_1(n)+T_2(n)=O(g_1(n) + g_2(n))$
- **Example**: $T_1(n)=O(n^2)$, $T_2(n)=O(n \log n)$ then $T_1(n)+T_2(n)=O(n^2 + n \log n) = O(n^2)$
- **Proof:** We know that there are constants n_1, n_2, c_1, c_2 **s.t.**
  - for every $n>n_1$ $T_1(n) < c_1 g_1(n)$. (definition of big-O)
  - for every $n>n_2$ $T_2(n) < c_2 g_2(n)$. (definition of big-O)
  - Now set $n'=\max\{ n_1, n_2 \}$, and $c'=c_1+c_2$, then
    - for every $n>n'$ we have that
    - $T_1(n)+T_2(n) < c_1 g_1(n) + c_2 g_2(n) \leq$
      $c' g_1(n) + c' g_2(n) =$
      $c' (g_1(n) + g_2(n))$

L1.62

More properties of big-O

•**Claim:** if $T_1(n)=O(g_1(n))$ and $T_2(n)=O(g_2(n))$ then
$T_1(n)\, T_2(n)=O(g_1(n)\, g_2(n))$

•**Example:** $T_1(n)=O(n^2)$, $T_2(n)=O(n \log n)$ then

$T_1(n)\, T_2(n)=O(n^3 \log n)$

•**Proof:** Homework

•Similar properties hold for Θ, Ω

L1.63