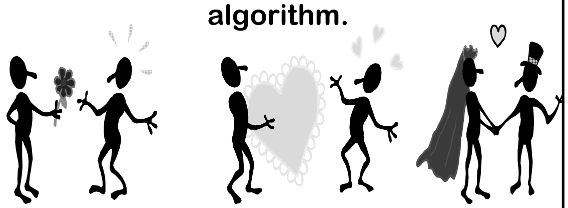
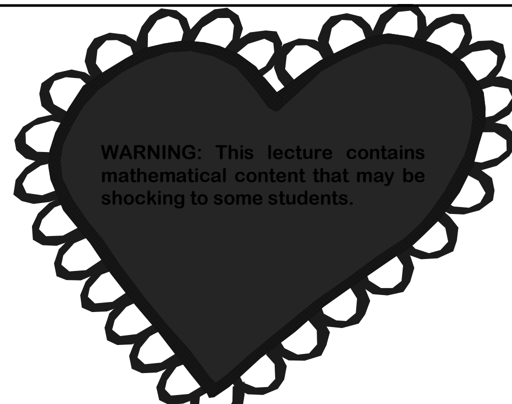
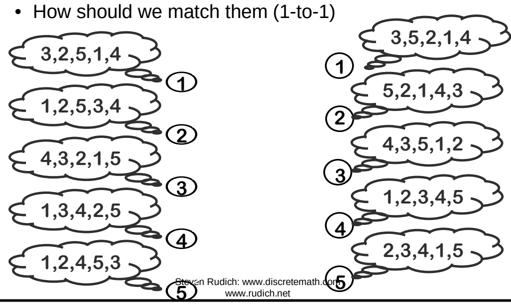
Credits:
Steven Rudich

The Mathematics Of 1950's Dating: Who wins the battle of the sexes? Stable marriage (matching) algorithm.





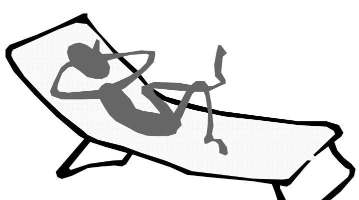
- There are *n* boys and *n* girls
- Each girl has her own ranked preference list of all the boys
  - Eg girl #1 highest prefs is boy #3 and lowest is girl #1
- Each boy has his own ranked preference list of the girls
- How should we match them (1-to-1)



There is more than one notion of what constitutes a "good" pairing.

- Maximizing total satisfaction
- Maximizing the minimum satisfaction
  - Western Europe
- Minimizing the maximum difference in mate ranks
  - Sweden
- Maximizing the number of people who get their first choice
  - Barbie and Ken Land

Etc etc



We will ignore the issue of what is "equitable"!

Rogue Couples

Suppose we pair off all the boys and girls. Now suppose that some boy and some girl prefer each other to the people to whom they are paired. They will be called a <u>rogue couple</u>.

They both would earn from dumping their mates and marry each other.

A matching is called <u>stable</u> if it contains no rogue couples.

<u>Rogue</u>: A vicious and restless animal, Elephant that leaves its herd

The study of stability will be the subject of the entire lecture.

We will:

- Analyze various mathematical properties of an algorithm that looks a lot like 1950's dating
- Discover the **mathematical truth** about which sex has the romantic edge

Given a set of preference lists, how do we find a stable pairing?

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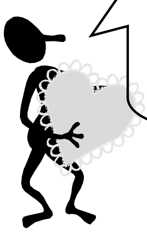
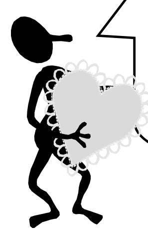


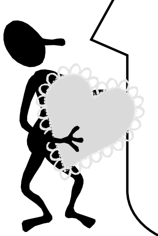
Wait! We don't even know that such a pairing always exists!

Given a set of preference lists, how do we find a stable pairing?



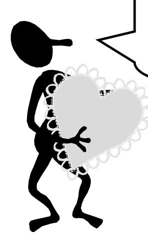
How could we change the question we are asking?

Better Questions:



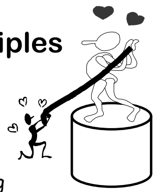
Will show: every set of preference lists have a stable pairing.
Will prove it by presenting a fast algorithm that, given any set of input lists, will output a stable pairing.

Idea: Allow the pairs to keep breaking up and reforming until they become stable.



Can you argue that the couples will not continue breaking up and reforming forever?

Terminology and principles



A boy can **propose** (a marriage) to a girl.

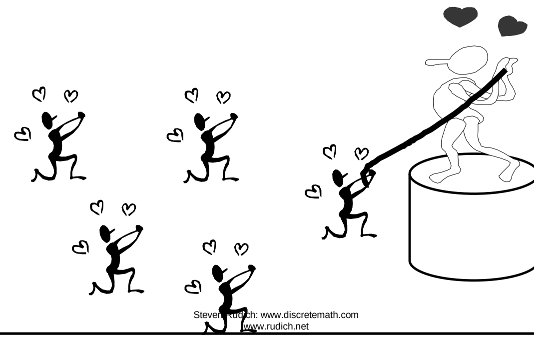
A girl can **reject** the proposal.

During most of the process, a girl would not accept a proposal, but either tell a proposing boy "maybe" . This is called "**putting the boy on a string**"

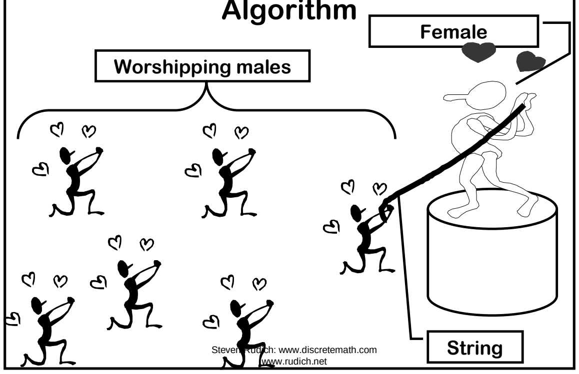
Once a boy is rejected, he **crosses off from his list** the rejecting girl – will not propose her again.

Once a boy proposes, he cannot change his mind until he is rejected.

The Traditional Marriage Algorithm



The Traditional Marriage Algorithm



Traditional Marriage Algorithm (TMA)

1) repeat{

- **Morning**
  - Each girl stands on her balcony
  - Each boy proposes under the balcony of the best girl whom he has not yet crossed off
- **Afternoon (for each girls with at least one proposal)**
  - To today's best offer: **"Maybe, come back tomorrow" (putting him on a string)**
  - To any other boy : **"No, I will never marry you"**
- **Evening**
  - Any rejected boy crosses the rejecting girl off his list.

}Until all boy are on strings.

2) Each girl marries the last boy she just said "maybe"

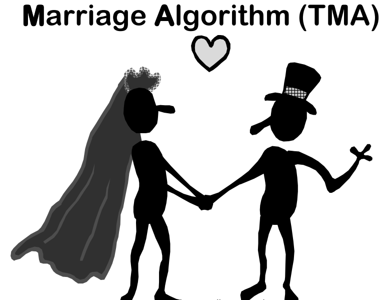
Note: Each boy proposes to girls in decreasing order on his list.

Lemma: **If a girl has a boy *b* on a string, then she will either marry him, or marry someone she prefers over him.**

Proof:

- She would only let go of *b* in order to "maybe" *b′* which she prefers over *b*
- She would only let go of *b′* for someone *b″* she prefers over *b′* etc.
- When the process terminates, she is left with someone she prefers over *b.*

Corollary: Each girl will marry her absolute favorite of the boys who visit her during the Traditional Marriage Algorithm (TMA)



Lemma: No boy can be rejected by all the girls

Proof by contradiction.

Suppose boy b is rejected by all the girls. At that point:

- Each girl must have a suitor other than ***b*** (By previous Lemma, once a girl has a suitor she will always have at least one)
- The ***n*** girls have ***n*** suitors, ***b*** not among them. Thus, there are at least ***n+1*** boys

Theorem: The TMA always terminates after at most n^2 days

- **Proof: The total length of the lists of all boys is $n \times n = n^2$.**
- **Each day at least one boy gets a "No", so at least one girl is deleted from one of the lists.**
- **Therefore, the number of days is bounded by the original size of the master list = n^2.**

Great! We know that TMA will terminate and produce a pairing.

But is it stable?

Theorem: Let T be the pairing produced by TMA. T is stable.

- Let b and g be any couple in T.
- Suppose b prefers g^* over g. We will argue that g^* prefers her husband over b.
- During TMA, boy b proposed to g^* before he proposed to g. Hence, at some point g^* rejected b for someone she preferred.
- By the Improvement lemma, the person she married was also preferable to b.
- Thus, every boy will be rejected by any girl he prefers to his wife. T is stable. QED.

Opinion Poll

Forget TMA for a moment

How should we define what we mean when we say "the optimal girl for boy b"?

Flawed Attempt:
"The girl at the top of b's list"

The Optimal Girl

A boy's optimal girl is the highest ranked girl for whom there is <u>some</u> stable matching in which the boy gets her.

(note – this is **not always** the highest girl on his list).

She is the best girl he can conceivably get in a stable world. Presumably, she might be better than the girl he gets in the stable pairing output by TMA.

The Pessimal boy

A girl's pessimal boy is the lowest ranked boy for whom there is <u>some</u> stable matching which the girl gets him.

He is the worst boy she can conceivably get in a stable world.

Dating Heaven and Hell

A stable matching is boys-optimal if every boy gets his optimal girl.

A matching is girls-pessimal if every girl gets her pessimal boy.

The Mathematical Truth!

The Traditional Marriage Algorithm yields a matching at which

- Each boy gets his optimal girl
- Each girl gets her pessimal boy

<u>Thm</u>: TMA produces a male-optimal pairing

- Suppose, for a contradiction, that some boy gets rejected by his optimal girl during TMA.
- Let t be the <u>earliest</u> time at which a boy b (**B**art) got rejected by his optimal girl g *(Gill)*
- *Gill* rejected *Bart* because she said "maybe" to a preferred boy b^* *(Stan).*
- *Stan* had not yet been rejected by his optimal girl (by the definition of t).
- Therefore,
  - *Gill* is either the optimal girl of *Stan*, or
  - *Gill* is better than the optimal girl of *Stan*.

Bart* got rejected by his optimal girl *Gill* because she said "maybe" to a preferred *Stan.

***Stan* likes *Gill* at least as much as his optimal girl.**

Let *S* be the matching at which *Bart* and *Gill* are married (*S* is NOT the result of the TMA)

Claim: *S* is not stable: the couple (*Stan* ,*Gill*) is a rogue couple (prefer each other over the ones they are married to)

Proof:

- *Stan* wants *Gill* more than his wife in *S*
  - *Gill* is as at least as good as his wife.
- *Gill* wants *Stan* more than her husband in *S*
  - *Bart* is her husband in *S* and she rejects him for *Stan* in TMA

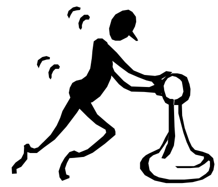
<u>Thm</u>: The TMA matching, *T*, is female-pessimal.

We know it is male-optimal. Suppose there is a stable pairing *S* where some girl *g* does worse than in *T*.
Let *b* be her husband in *T*.
Let *b** be her husband in *S*.

- By assumption, *g* likes *b* better than her husband in *S*
- *b* likes *g* better than his wife in *S*
  - (since *g* is his optimal girl)
- So (*g,b*) is a rogue couple.
- **Therefore, *S* is not stable.**

Contradiction

Advice to females



Learn to make the first move.

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Dan Gusfield and Robert W. Irving, *The Stable Marriage Problem: Structures and Algorithms*, MIT Press, 1989