CSc 573 **Take-Home Exam 1** **Spring 2007**

This exam is take-home, and is due back at the start of class on Thursday, March 8. The exam consists of three problems worth a total of 50 points.

Do Problem (1), and then *choose two questions* from among Problems (2) through (4). In other words, do a total of three questions. If you do more than three questions, only three will be graded.

When writing your solutions, use only one side of a sheet of paper. Start each problem on a new page. Be neat and concise.

You may use theorems proved in class, the homework, or the book — *except for* ones you are asked to prove — as long as you cite them.

If you are in doubt about the interpretation of a problem, clearly state your assumptions (which should be reasonable) in your solution.

You are not allowed to receive help from classmates or friends — *individual work is expected*. With a take-home exam you have the opportunity to demonstrate your understanding of the material without the pressure of an in-class time limit. On my part, this requires that I respect your ability to follow an honor system. Please earn this respect.

The exam should not take more than six hours, but probably won't take less than three hours. Work steadily, and be sure to allocate enough time.

Good luck!

(1) **(Self reducibility)** (10 points) In an undirected graph G, a cycle is *Hamiltonian* if it visits every vertex of G exactly once. The Undirected Hamiltonian Cycle Problem is to decide the language

$$\text{UndirHamCycle} := \left\{ \langle G \rangle \;:\; \begin{pmatrix} G \text{ is an undirected graph that} \\ \text{contains a Hamiltonian cycle} \end{pmatrix} \right\}.$$

Suppose that UndirHamCycle can be *decided* in polynomial time. Show that in this case, one can *find* a Hamiltonian cycle, if one exists, in polynomial time.

(Hint: Make repeated calls to a decision algorithm for UndirHamCycle.)

(2) **(Successive zeroes)** (20 points) An $m \times n$ matrix A whose entries are 0's or 1's has the *successive zeroes property* if the columns of A can be reordered such that in every row i of the reordered matrix, the 0's in row i are all in successive columns. (In other words, after reordering its columns, no 1's appear between 0's in its rows.) Let

$$\text{SuccZeroes} := \left\{ \langle A, k \rangle \;:\; \begin{pmatrix} A \text{ is an } m \times n \text{ matrix of 0's and 1's that contains an} \\ m \times k \text{ submatrix with the successive zeroes property} \end{pmatrix} \right\}.$$

Prove that SuccZeroes is NP-complete.

(Hint: Use a reduction from the Undirected Hamiltonian Path Problem, which is to decide the language

$$\text{UndirHamPath} := \left\{ \langle G, s, t \rangle \;:\; \begin{pmatrix} G \text{ is an undirected graph that} \\ \text{contains a Hamiltonian } (s,t)\text{-path} \end{pmatrix} \right\}.$$

(3) **(Sum split)** (20 points) Given a multiset $X = \{x_1, x_2, \ldots, x_n\}$ of integers, a *sum split* of X is a multisubset $Y \subseteq X$ such that

$$\sum_{y \in Y} y \;=\; \sum_{x \in X - Y} x.$$

(In other words, the sum of the numbers in Y is the same as the sum of the numbers not in Y.) In the above sums, a value appears in a term each time it appears in the multiset. Also, the multisubset Y can contain an element of X at most the number of times it appears in X. Let

$$\text{SumSplit} \;:=\; \left\{ \langle X \rangle \;:\; X \text{ is a multiset of integers that contains a sum split} \right\}.$$

Prove that SumSplit is NP-complete.

(Hint: Use a reduction from the Subset Sum Problem, which is to decide the language

$$\text{SubsetSum} \;:=\; \left\{ \langle S, k \rangle \;:\; \begin{pmatrix} S \text{ is a multiset of integers that contains} \\ \text{a multisubset whose values sum to } k \end{pmatrix} \right\}.$$

In SubsetSum, the multisubset can contain an element of S at most the number of times it appears in S.)

(4) **(Cycle set)** (20 points) In a directed graph $G = (V, E)$, a *cycle set* is a vertex subset $S \subseteq V$ such that S contains at least one vertex from every cycle in G. Let

$$\text{CycleSet} \;:=\; \left\{ \langle G, k \rangle \;:\; \begin{pmatrix} G \text{ is a directed graph that contains} \\ \text{a cycle set of at most } k \text{ vertices} \end{pmatrix} \right\}.$$

Prove that CycleSet is NP-complete.

(Hint: Use a reduction from the Vertex Cover Problem, which is to decide the language

$$\text{VertexCover} \;:=\; \left\{ \langle G, k \rangle \;:\; \begin{pmatrix} G \text{ is an undirected graph that contains} \\ \text{a vertex cover of at most } k \text{ vertices} \end{pmatrix} \right\}.$$

Recall that a *vertex cover* of an undirected graph G is a vertex subset C such that every edge of G has at least one endpoint in C.)