CSc 573 **Take-Home Exam 2** **Spring 2007**

This exam is take-home, and is due back at the start of class on Thursday, April 26. The exam consists of three problems worth a total of 50 points.

When writing your solutions, use only one side of a sheet of paper. Start each problem on a new page. Be neat and concise.

You may use theorems proved in class, the homework, or the book — *except for* ones you are asked to prove — as long as you cite them.

If you are in doubt about the interpretation of a problem, clearly state your assumptions (which should be reasonable) in your solution.

You are not allowed to receive help from classmates or friends — *individual work is expected*. With a take-home exam you have the opportunity to demonstrate your understanding of the material without the pressure of an in-class time limit. On my part, this requires that I respect your ability to follow an honor system. Please earn this respect.

The exam should not take more than six hours, but probably won't take less than three hours. Work steadily, and be sure to allocate enough time.

Good luck!

(1) **(Get home)** (20 points) Consider the following two-player game, called the *Get Home Game*, played by players I and II. Get Home is played on an *undirected* graph $G = (V, E)$, with three special vertices $s, x, y \in V$ that are all distinct. Vertex s is the *start vertex*, vertex x is the *home vertex* for player I, and vertex y is the *home vertex* for player II.

Play starts from node s with player I moving first. Moves alternate between players I and II. Each move extends an undirected path P from s by choosing an edge $e \in E$ that is incident to the last node on P. The chosen edge e must *not* go to a vertex already visited by P.

Player I wins if the path reaches his home vertex x, or if player II has the move and cannot extend the path. Player II wins if the path reaches his home vertex y, or if player I has the move and cannot extend the path.

Let

$$\text{GetHome} := \left\{ \langle G, s, x, y \rangle \ : \ \begin{pmatrix} \text{Player I has a winning strategy in the Get} \\ \text{Home Game played on undirected graph } G \\ \text{with start vertex } s \text{ and home vertices } x, y \end{pmatrix} \right\}.$$

Prove that GetHome is PSpace-complete.

(2) **(Reachability-preserving subgraph)** (20 points) For a directed graph $G = (V, E)$, the subgraph $\widetilde{G} = (V, F)$ defined by edge subset $F \subseteq E$ is an *reachability-preserving subgraph* iff for all ordered pairs (s, t) of distinct vertices $s, t \in V$, graph G has an (s, t)-path iff subgraph $\widetilde{G}$ has an (s, t)-path. In other words, a reachability-preserving subgraph maintains reachability between all pairs of vertices, while possibly removing edges.

Let

$$\text{ReachPreserve} := \left\{ \langle G, k \rangle \ : \ \begin{pmatrix} G \text{ is a directed graph that contains a reachabil-} \\ \text{ity-preserving subgraph with at most } k \text{ edges} \end{pmatrix} \right\}.$$

Prove that ReachPreserve is NP-complete.

(Note: Use a reduction from one of the problems that we proved NP-complete in the lectures, namely ThreeSat, Clique, VertexCover, SubsetSum, DirectedHamiltonianPath, or UndirectedHamiltonianPath.)

(3) **(Supersequence)** (10 points) String x is a *supersequence* of string y if string y can be formed from string x by deleting some of the letters in string x. So for example, the string `shelter` is a supersequence of the string `set`.

Let

$$\text{NoSupersequence} := \left\{ \langle M, y, k \rangle : \left(\begin{array}{l} M \text{ is a DFA such that } L(M) \text{ does } \textit{not} \text{ contain} \\ \text{a supersequence } x \text{ of string } y \text{ with } |x| \leq k \end{array} \right) \right\}.$$

Prove that NoSupersequence $\in$ PSpace.

(Note: To run in space that is polynomial in the length of the input, the amount of space used by the decider for NoSupersequence must be polynomial in $\log k$, since k is an integer that is encoded in binary in the input.)