Continued Fractions

Continued fractions are part of the "lost mathematics," the mathematics now considered too advanced for high school and too elementary for college.

— Petr Beckmann, *A History of Pi*

Most persons taking courses in mathematics do not encounter continue fractions. When first encountered, they have a forbidding appearance. Yet continued fractions have an elegant theory and are important in several branches of mathematics.

A continued fraction is a fraction in which the numerators and denominators may contain (continued) fractions. Displayed in their full laddered form, they look like this:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}}}}$$

See Figure Ω.1 on the next page for other examples.

The numerators and denominators in a continued fraction can themselves be complicated, as evidenced by Figure Ω.1i. Most work on continued fractions deals with *ordinary* continued fractions, in which the numerators and denominators are numbers:

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cfrac{b_4}{a_5 + \cfrac{b_5}{a_6 + \cfrac{b_6}{a_7 + \cfrac{b_7}{a_8 + \dots}}}}}}}$$

Two sequences completely characterize an ordinary continued fraction: a_1, a_2, a_3, a_4, … and b_1, b_2, b_3, b_4 … .

A *simple* continued fraction is an ordinary continued fraction in which all the numerators are 1 and all the denominators are integers and positive except possibly a_1:

a

$$e-1=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\dots}}}}}}}}$$

b

$$\frac{1}{e-2}=1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\cfrac{5}{6+\cfrac{6}{7+\cfrac{7}{8+\dots}}}}}}}$$

c

$$\frac{\pi}{2}=1-\cfrac{1}{3-\cfrac{2\cdot 3}{1-\cfrac{1\cdot 2}{3-\cfrac{4\cdot 5}{1-\cfrac{3\cdot 4}{3-\cfrac{6\cdot 7}{1-\cfrac{5\cdot 6}{3-\dots}}}}}}}$$

d

$$\frac{4}{\pi}=1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cfrac{9^2}{2+\cfrac{11^2}{2+\cfrac{13^2}{2+\dots}}}}}}}$$

e

$$\sin(x)=\cfrac{x}{1+\cfrac{x^2}{(2\cdot 3-x^2)+\cfrac{2\cdot 3x^2}{(4\cdot 5-x^2)+\cfrac{4\cdot 5x^2}{(6\cdot 7-x^2)+\dots}}}}$$

f

$$\tan(1)=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{7+\cfrac{1}{1+\dots}}}}}}}}$$

g

$$\frac{1+\sqrt{5}}{2}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}}}}}$$

h

$$\log(1+x)=\cfrac{x}{1+\cfrac{1^2x}{2+\cfrac{1^2x}{3+\cfrac{2^2x}{4+\cfrac{2^2x}{5+\cfrac{3^2x}{6+\cfrac{3^2x}{7+\dots}}}}}}}$$

i

$$\sqrt[3]{2}+2=\cfrac{1}{3+\cfrac{3+\cfrac{1}{3+\cfrac{3+\dots}{3+\dots}}}{3+\cfrac{3+\cfrac{1}{3+\cfrac{3+\dots}{3+\dots}}}{3+\cfrac{3+\cfrac{1}{3+\cfrac{3+\dots}{3+\dots}}}{3+\cfrac{3+\cfrac{1}{3+\cfrac{3+\dots}{3+\dots}}}{3+\cfrac{3+\cfrac{1}{3+\cfrac{3+\dots}{3+\dots}}}{3+\cfrac{3+\cfrac{1}{3+\cfrac{3+\dots}{3+\dots}}}{3+\dots}}}}}}}$$

Figure Ω.1. A Gallery of Continued Fractions

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + \cfrac{1}{a_7 + \cfrac{1}{a_8 + \ldots}}}}}}}$$

Only one sequence is needed to characterize a simple continued fraction. For example, the continued-fraction sequence for π is

3, 7, 15, 1, 292, 1, 1, 1, … .

As you'd expect, this sequence is infinite.

There are several important facts about simple continued-fraction sequences:

1. Rational numbers (fractions) have finite sequences. An example is 11/13, which has the sequence 0, 1, 5, 2.
2. Irrational numbers have infinite sequences.
3. Quadratic irrationals have periodic sequences. An example is $\sqrt{7}$, which has the sequence $2, \overline{1, 1, 1, 4}$.
4. All other irrational numbers have non-periodic sequences. The sequence for π, shown above, is an example.
5. There is a one-to-one correspondence between an irrational number and its simple continued-fraction sequence. Furthermore, any periodic sequence of positive integers represents a unique irrational number. (For rational numbers, there are two equivalent sequences: one that ends … a_m, 1 and one that ends … $a_m - 1$.)

Computing Continued Fractions

Continued fractions are closely related to the familiar Euclidean algorithm for computing the greatest common divisor of two integers, *i* and *j*. Euclid's algorithm might look like this in pseudo-code:

```
until j = 0 do {
  r := remdr(i, j)
  i := j
  j := r
  }
print(i)          # previous value of j
```

The terms in the simple continued faction for i / j consist of values of $i \div j$ (integer division, remainder discarded) in the loop above:

```
until j = 0 do {
 print(i ÷ j)
 r := remdr(i, j)
 i := j
 j := r
 }
```

The problem with trying to compute continued fractions for irrational numbers is that floating-point numbers used by computers to represent real numbers are finite approximations to real numbers, and hence they really are rational numbers whose values are "close" to the corresponding real numbers. For example, the standard 64-bit floating-point encoding for π is

$$7074237752028440/2^{51}$$

The corresponding continued-fraction sequence is, of course, finite:

3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 3, 3, 2,
1, 3, 3, 7, 2, 1, 1, 3, 2, 42, 2

and only the first 13 terms are the same as for the sequence for the actual irrational number:

3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1,
2, 2, 2, 2, 1, 84, 2, 1, …

Patterns

Simple continued-fraction sequences for rational numbers usually are short and any patterns are accidental and mostly uninteresting.

Since quadratic irrationals have periodic simple continued-fraction sequences, they have patterns that may be of interest in designing weaves.

Simple continued-fraction sequences for other irrationals are not periodic and most have no evident patterns.

Some, however, do. An example is tan(1) (see Figure Ω.1f), whose simple continued-fraction sequence is

$$1, \overline{1, 2n+1} \qquad n = 1, 2, 3, \ldots$$ Explain overbar notation.

Another example is $e-1$ (see Figure Ω.1a), whose simple continued-fraction sequence is

$1, \overline{1, 2n, 1}$ $\quad\quad$ $n = 1, 2, 3, \ldots$

Such sequences have periodic *forms*. The simple continued-fraction sequence for π has no such structure, but there is an ordinary continued-fraction for π/4 (see Figure Ω.1d) that has numerator and denominator sequences with periodic forms:

numerators: $\overline{(2n-1)^2}$ $\quad$ $n = 1, 2, 3, \ldots$

denominators: $1, \overline{2}$

Figures Ω.2 through Ω.4 indicate some possibilities for weaves based on continued fractions..

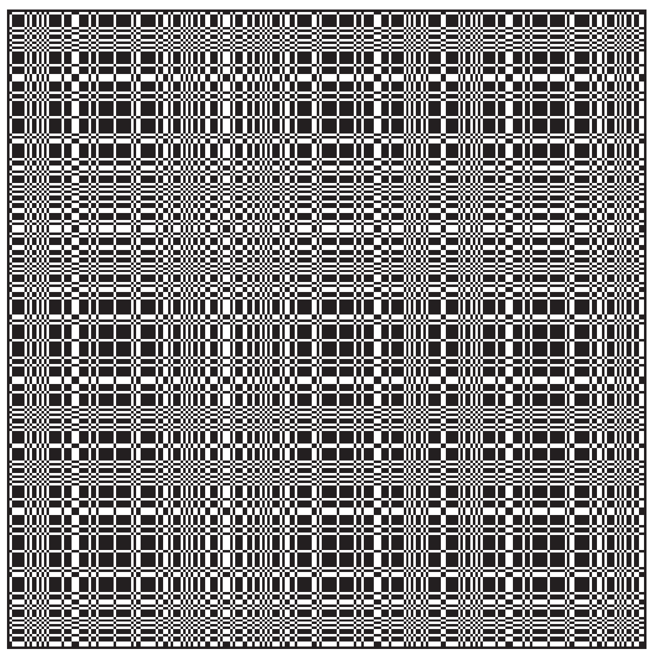

Figure Ω2. $\sqrt{10089}$**, Tabby Tie-Up**

Need ideas for better examples.

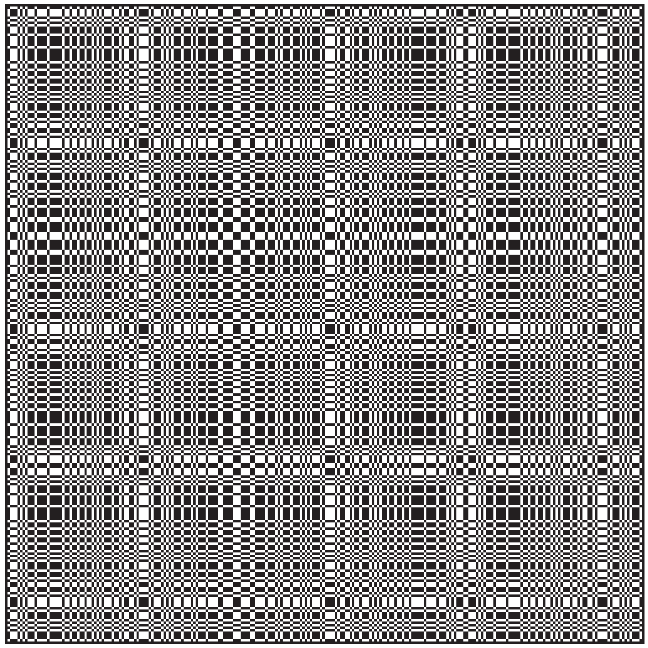

Figure Ω.3. $\sqrt{9949}$, Tabby Tie-Up

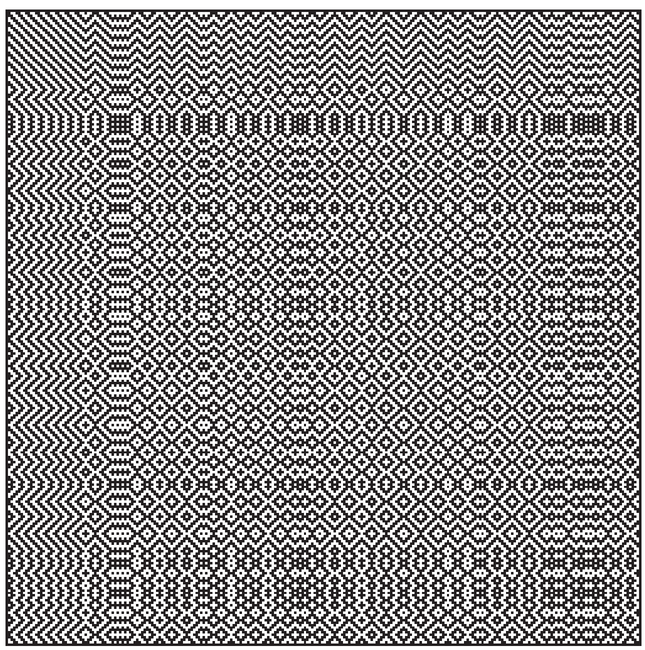

Figure Ω4. $\sqrt{9949}$, Twill Tie-Up

Need more discussion of designing with continued fractions.

Learning More About Continued Fractions

Much of the literature about continued fractions is highly technical and specialized. There are, however, a few books that are accessible [1-3]. There also are Web resources [4-5].