# CSc 466/566

# Computer Security

# 8 : Cryptography — Digital Signatures

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# Outline

1. Introduction
2. RSA Signature Scheme
3. Elgamal Signature Scheme
4. Cryptographic Hash Functions
5. Birthday attacks
6. Summary

# Digital Signatures

- In this lecture we are going to talk about cryptographic hash functions (checksums) and digital signatures.
- We want to be able to
  1. Detect tampering: is the message we received the same as the message that was sent?
  2. Authenticate: did the message come from who we think it came from?

# Digital Signatures...

- More specifically, we want to ensure:
  1. Nonforgeability: Eve should not be able to create a message that appears to come from Alice.
  2. Nonmutability: Eve should not be able to take a valid signature for one message from Alice, and apply it to another one.
  3. Nonrepudiation: Alice should not be able to claim she didn't sign a document that she did sign.

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2. Alice sends $M$ and the signature $S_{\mathrm{Alice}}(M)$ to Bob.
3. Bob decrypts the document using Alice's public key, thereby verifying her signature.

# Digital Signatures...

- This works because for many public key ciphers

$$
\begin{aligned}
D_{S_B}(E_{P_B}(M)) &= M \\
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\end{aligned}
$$

  i.e. we can reverse the encryption/decryption operations.

- That is, Bob can apply the decryption function to a message with his private key $S_B$, yielding the signature $\mathrm{sig}$:

$$\mathrm{sig} \leftarrow D_{S_B}(M)$$

- Then, anyone else can apply the encryption function to $\mathrm{sig}$ to get the message back. Only Bob (who has his secret key) could have generated the signature:

$$E_{P_B}(\mathrm{sig}) = M$$

# Digital Signatures. . .

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Alice

$\boxed{M}$

Bob sent $M$?

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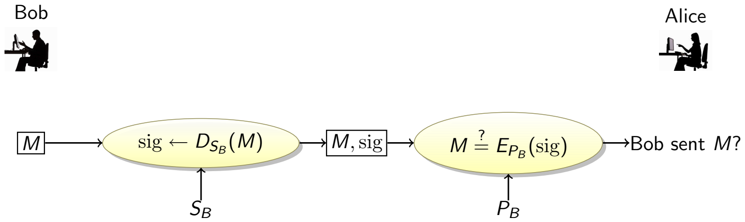

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  2. Verify that $M \stackrel{?}{=} S^e \bmod n$.

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- This is equivalent to being able to break RSA encryption.

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- Not usually a problem since we normally sign hashes.

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  1. Pick a prime $p$.
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  3. Pick a random number $x$ between 1 and $p - 2$.
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  1. Verify $y^a \cdot a^b \bmod p \stackrel{?}{=} g^M \bmod p$.

# Elgamal Signature Algorithm: Correctness

- We have:

$$\begin{aligned} y &= g^x \bmod p \\ a &= g^k \bmod p \\ b &= k^{-1}(M - xa) \bmod (p-1) \end{aligned}$$

- Show that $y^a \cdot a^b \bmod p = g^M \bmod p$.

$$\begin{aligned} y^a a^b \bmod p &= (g^x \bmod p)^a((g^k \bmod p)^{(}k^{-1}(M - xa) \bmod (p-1)) \text{ m} \\ &= g^{xa} g^{kk^{-1}(M-xa) \bmod (p-1)} \bmod p \\ &= g^{xa} g^{(M-xa) \bmod (p-1)} \bmod p \\ &= g^{xa} g^{M-xa} \bmod p \\ &= g^{xa+M-xa} \bmod p \\ &= g^M \bmod p \end{aligned}$$

# Elgamal Signature Algorithm: Security

- We have:

$$\begin{aligned} y &= g^x \bmod p \\ a &= g^k \bmod p \\ b &= k^{-1}(M - xa) \bmod (p-1) \end{aligned}$$

- $k$ is random $\Rightarrow$ $b$ is random!
- To the adversary, $b$ looks completely random.
- The adversary must compute $k$ from $a = g^k \bmod p \Leftrightarrow$ compute discrete log!
- If Alice reuses $k \Rightarrow$ The adversary can compute the secret key.

# Outline


1. Introduction
2. RSA Signature Scheme
3. Elgamal Signature Scheme
4. Cryptographic Hash Functions
5. Birthday attacks
6. Summary

# Cryptographic Hash Functions

- Public key algorithms are too slow to sign large documents. A better protocol is to use a one way hash function also known as a cryptographic hash function (CHF).
- CHFs are checksums or compression functions: they take an arbitrary block of data and generate a unique, short, fixed-size, bitstring.

```
> echo "hello" | sha1sum
f572d396fae9206628714fb2ce00f72e94f2258f  -
> echo "hella" | sha1sum
1519ca327399f9d699afb0f8a3b7e1ea9d1edd0c  -
> echo "can't believe it's not butter!"|sha1sum
34e780e19b07b003b7cf1babba8ef7399b7f81dd  -
```

# Signature Protocol

1. Bob computes a one-way hash of his document.

$$\begin{aligned} \text{hash} &\leftarrow h(M) \\ \text{sig} &\leftarrow E_{S_B}(\text{hash}) \\ D_{P_B}(\text{sig}) &\stackrel{?}{=} h(M) \end{aligned}$$

# Signature Protocol

1. Bob computes a one-way hash of his document.
2. Bob encrypts the hash with his private key, thereby signing it.

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3. Bob sends the encrypted hash and the document to Alice.

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# Signature Protocol

1. Bob computes a one-way hash of his document.
2. Bob encrypts the hash with his private key, thereby signing it.
3. Bob sends the encrypted hash and the document to Alice.
4. Alice decrypts the hash Bob sent him, and compares it against a hash she computes herself of the document. If they are the same, the signature is valid.

$$
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D_{P_B}(\text{sig}) &\stackrel{?}{=} h(M)
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# Signature Protocol. . .

Bob

Alice

$\boxed{M}$

Bob sent $M$?

- Advantage: the signature is short; defends against MITM attack.

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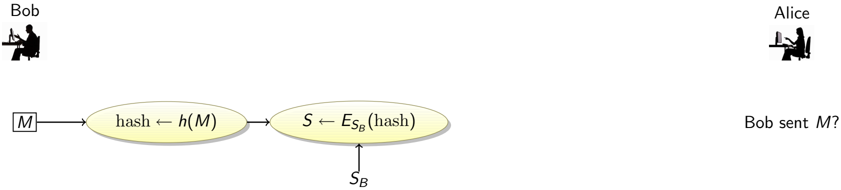
# Signature Protocol. . .

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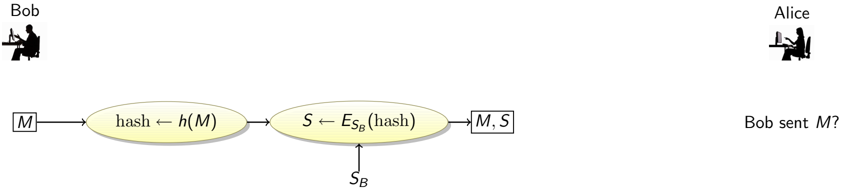
# Signature Protocol. . .

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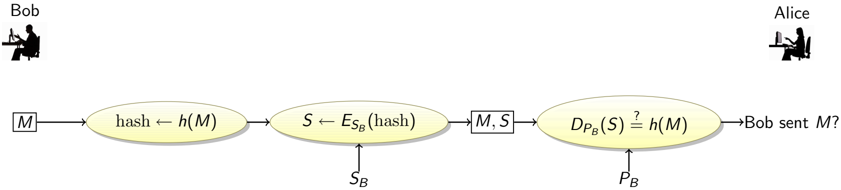


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# Cryptographic Hash Functions. . .

- CHFs should be
  1. deterministic
  2. one-way
  3. collision-resistant

  i.e., easy to compute, but hard to invert.
- I.e.
  - given message $M$, it's easy to compute $y \leftarrow h(M)$;
  - given a value $y$ it's hard to compute an $M$ such that $y = h(M)$.

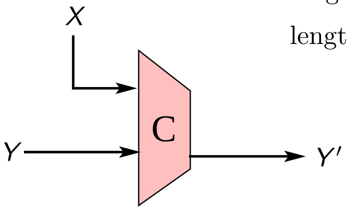
  This is what we mean by CHFs being one-way.

# Weak vs. Strong Collision Resistance

- CHFs also have the property to be collision resistant.
- Weak collision resistance:
  - Assume you have a message $M$ with hash value $h(M)$.
  - Then it should be hard to find a different message $M'$ such that $h(M) = h(M')$.
- Strong collision resistance:
  - It should be hard to find two different message $M_1$ and $M_2$ such that $h(M_1) = h(M_2)$.
- Strong collisions resistance is hard to prove.

# Merkle-Damgøard Construction

- Hash functions are often built on a compression function $C(X, Y)$:

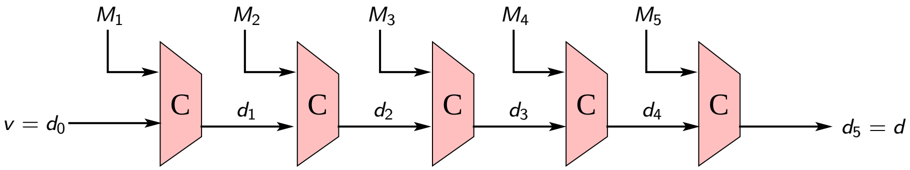



length$(X)=m$

length$(Y)=$length$(Y')=n$

- $X$ is (a piece of) the message we're hashing.
- $Y$ and $Y'$ is the hash value we're computing.

# Merkle-Damgøard Construction...




- For long messages $M$ we break it into pieces $M_1, \ldots, M_k$, each of size $m$.
- Our initial hash value is an initialization vector $v$.
- We then compress one $M_i$ at a time, chaining it together on the previous hash value.

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# The Birthday Problem

- Given a group of $n$ people, what is the probability that two share a birthday?
- Examine the probability that no two share a birthday: (let $B_i$ be person i's birthday)
  - $n = 1 : 1$
  - $n = 2 : 364/365$
  - $n = 3$ : probability that $B_3$ differs from both $B_1$ and $B_2$ and that none of the first two share a birthday: $363/365 * 364/365$
  - $n = 4$ :, probability that $B_4$ differs from all of $B_{1...3}$ and that none of the first three share a birthday: $362/365 * (363/365 * 364/365)$
  - and so on ...

# The Birthday Problem

- This generalizes to

$$\frac{365!}{365^n(365-n)!}$$

- It takes only 23 people to give greater than .5 probability that two people share a birthday in a domain with cardinality 365.
- For a domain with cardinality $c$, .5 probability is reached with approximately $1.2\sqrt{c}$ numbers.
- So what does this have to do with checksums?

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- Wrong! By the birthday paradox, it is likely that two messages will have the same hash value!
- Security is $\approx 2^{b/2}$ not $2^b$.
- Thus, a hash-function with 256-bit output has 128-bit security.

# Birthday Attacks

- Little Billy wants to be the sole beneficiary of Grandma's will
- He prepares two message templates, like the one Charlie made, one being a field trip permission slip, and the other being a will in which Grandma bequeaths everything to her sweet grandson.
- Little Billy finds a pair of messages, one generated from each template, with equal checksums
- Little Billy has Grandma sign the field trip permission slip
- Little Billy now has a signature that checks out against the will he created
- Profit!!

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# Summary

- Digital signatures make a message tamper-proof and give us authentication and nonrepudiation
- They only show that it was signed by a specific key, however
- It's cheaper to sign a checksum of the message rather than the whole message
  - Cryptographic checksums are necessary to do this securely

# Readings and References

- Chapter 8.1.7, 8.2.1, 8.5.2 in *Introduction to Computer Security*, by Goodrich and Tamassia.

# Acknowledgments

Additional material and exercises have also been collected from these sources:

1. Matthew Landis, *620—Fall 2003—Cryptographic Checksums and Digital Signatures*.
2. RFC1321 (MD5), www.ietf.org/rfc/rfc1321.txt