CSc 520

Principles of Programming Languages

22: Lambda Calculus — Reductions

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Lambda Reductions

- To evaluate a lambda expression we reduce it until we can apply no more reduction rules. There are four principal reductions that we use:
  1. α-reduction —variable renaming to avoid name clashes in β-reductions.
  2. β-reduction —function application.
  3. η-reduction —formula simplification.
  4. δ-reduction —evaluation of predefined constants and functions.

α-reduction — Introductory Example

- Blindly applying β-reductions can lead to a problem known as variable capture. To prevent this we need a way to change the name of a variable.
- Here's an example of an α-reduction:

$$((\lambda y.(\lambda f.(f \quad x))) \quad y)$$
$$\Downarrow\alpha$$
$$((\lambda z.(\lambda f.(f \quad x))) \quad z)$$

β-reduction — Introductory Example

- The equivalence of function application in a functional language is called β-reduction.
- Here's an example of using β-reductions to evaluate a lambda expression:

$$\text{Twice} \equiv (\lambda f.(\lambda x.(f \quad (f \quad x))))$$

$$((\text{Twice} \quad (\lambda n.(\text{add} \quad n \quad 1))) \quad 5) \Rightarrow$$

$$(((\lambda f.(\lambda x.(f \quad (f \quad x)))) \quad (\lambda n.(\text{add} \quad n \quad 1))) \quad 5) \Rightarrow_\beta$$

$$((\lambda x.((\lambda n.(\text{add} \quad n \quad 1)) \quad ((\lambda n.(\text{add} \quad n \quad 1)) \quad x))) \quad 5) \Rightarrow_\beta$$

$$((\lambda n.(\text{add} \quad n \quad 1)) \quad ((\lambda n.(\text{add} \quad n \quad 1)) \quad 5)) \Rightarrow_\beta$$

-reduction — Introductory Example. . .

$((\text{Twice}\ \ (\lambda n.(\text{add}\ \ n\ \ 1)))\ \ 5) \Rightarrow$

$(((\lambda f.(\lambda x.(f\ \ (f\ \ x))))\ \ (\lambda n.(\text{add}\ \ n\ \ 1)))\ \ 5) \Rightarrow_\beta$

$((\lambda x.((\lambda n.(\text{add}\ \ n\ \ 1))\ \ ((\lambda n.(\text{add}\ \ n\ \ 1))\ \ x)))\ \ 5) \Rightarrow_\beta$

$((\lambda n.(\text{add}\ \ n\ \ 1))\ \ ((\lambda n.(\text{add}\ \ n\ \ 1))\ \ 5)) \Rightarrow_\beta$

$(\text{add}\ \ ((\lambda n.(\text{add}\ \ n\ \ 1))\ \ 5)\ \ 1) \Rightarrow_\beta$

$(\text{add}\ \ (\text{add}\ \ 5\ \ 1)\ \ 1)$

- No further β-reductions can be made.

δ-reduction — Introductory Example

- A δ-reduction is used to evaluate non-pure lambda expression, i.e. those that contain predefined constants and functions (such as add, etc.).
- Without δ-reductions we couldn't evaluate the previous example any further:

$((\text{Twice}\ \ (\lambda n.(\text{add}\ \ n\ \ 1)))\ \ 5) \Rightarrow_*$

$(\text{add}\ \ (\text{add}\ \ 5\ \ 1)\ \ 1) \Rightarrow_\delta$

$(\text{add}\ \ 6\ \ 1) \Rightarrow_\delta$

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Free Variables

Free Variable Substitution

- To work with α-reductions we first need to define the concept of a variable substitution.
- The notation

$$E[v \rightarrow M]$$

means "replace all free occurences of v with M in the expression E."

- Example:

$$(\lambda v.(\text{mul}\ y\ v))[y \rightarrow z] = (\lambda v.(\text{mul}\ z\ v))$$

Free Variable Substitution. . .

- A more traditional notation (used by Scott, for example) for
$$E[v \rightarrow M]$$
is
$$E[M\backslash v]$$
- Example:
$$(\lambda v.(\text{mul } y\ v))[z\backslash y] = (\lambda v.(\text{mul } z\ v))$$

Free Variable Substitution

- Variable replacements aren't legal in
$$E[v \rightarrow M]$$
if a free variable in M becomes bound.
- For example, the substitution
$$(\lambda x.(\text{mul } y\ x))[y \rightarrow x] \Rightarrow (\lambda x.(\text{mul } x\ x))$$
is illegal because it causes a change in semantics: the new expression represents a squaring operation, the original doesn't.
- This is known as variable capture or a name clash.

Computing Free Variables

- The free variables of an expression E, FV(E) is computed as
  1. FV$(c) = \emptyset$ for any constant c.
  2. FV$(x) = \{x\}$ for any variable x.
  3. FV$((E_1\ E_2))$ = FV$(E_1) \cup$ FV(E_2).
  4. FV$((\lambda x.E))$ = FV$(E) - \{x\}$.
- An expression that has no free variables (FV$(E) = \emptyset$) is called closed.

Computing Free Variables — Example

$$\text{FV}((\lambda x.(y\ \ (\lambda y.((y\ \ x)\ \ z))))) =$$
$$\text{FV}((y\ \ (\lambda y.((y\ \ x)\ \ z)))) - \{x\} =$$
$$(\text{FV}(y) \cup \text{FV}((\lambda y.((y\ \ x)\ \ z)))) - \{x\} =$$
$$(\{y\} \cup (\text{FV}(((y\ \ x)\ \ z)) - \{y\})) - \{x\} =$$
$$(\{y\} \cup ((\text{FV}((y\ \ x)) \cup \text{FV}(z)) - \{y\}) - \{x\} =$$
$$(\{y\} \cup ((\text{FV}(y) \cup \text{FV}(x)) \cup \{z\}) - \{y\}) - \{x\} =$$
$$(\{y\} \cup ((\{y\} \cup \{x\}) \cup \{z\}) - \{y\}) - \{x\} =$$
$$\{y, z\}$$

Variable Substitution — Algorithm

a) $v[v \to E_1] = E_1$ for any variable v

b) $x[v \to E_1] = x$ for any variable $x \neq v$

c) $c[v \to E_1] = c$ for any constant c

d) $(E_1\ E_2)[v \to E_3] = (E_1[v \to E_3]\ E_2[v \to E_3])$

e) $(\lambda v.E)[v \to E_1] = (\lambda v.E)$

f) $(\lambda x.E)[v \to E_1] = (\lambda x.E[v \to E_1])$, when $x \neq v$ and $x \notin \text{FV}(E_1)$

g) $(\lambda x.E)[v \to E_1] = (\lambda z.E[x \to z][v \to E_1])$, when $x \neq v$ and $x \in \text{FV}(E_1)$ where $z \neq v$ and $z \notin \text{FV}((E\ E_1))$

- In g) the first substitution $E[x \to z]$ replaces a bound variable x by a new bound variable z.

Variable Substitution — Example

$$(\lambda y.((\lambda f.(f\ \ x))\ \ y))[x \to (f\ \ y)] \xRightarrow{\text{g)}}$$

$$(\lambda z.((\lambda f.(f\ \ x))\ \ y)[y \to z][x \to (f\ \ y)]) =$$

$$(\lambda z.((\lambda f.(f\ \ x))\ \ z)[x \to (f\ \ y)]) \xRightarrow{\text{d)}}$$

$$(\lambda z.((\lambda f.(f\ \ x))[x \to (f\ \ y)]\ \ z[x \to (f\ \ y)])) \xRightarrow{\text{b)}}$$

$$(\lambda z.((\lambda g.(g\ \ x))[x \to (f\ \ y)]\ \ z)) \xRightarrow{\text{d,b,a)}}$$

$$(\lambda z.((\lambda g.(g\ \ (f\ \ y)))\ \ z))$$

Reductions

Reductions

- The main rule for evaluating a lambda expression is called β-reduction.
- β-reduction is similar to function application in a functional language.
- To prevent variable capture (when a free variable in E_1 becomes bound during the substitution $E[v \to E_1]$), we also need α-reductions.
- δ-reductions are used to evaluate predefined functions such as add.
- η-reductions are not strictly necessary but can be used to clean up messy expressions.

α-reductions

- Scott calls this α-conversion.
- α-reduction says that we can replace v by w in $(\lambda v.E)$ as long as w does not occur free in E.
- Formally,
$$(\lambda v.E) \Rightarrow_\alpha (\lambda w.E[v \to w])$$
where w does not occur free in E.
- Example:
$$(\lambda y.((\lambda f.(f\ x))\ y)) \Rightarrow_\alpha (\lambda z.((\lambda f.(f\ x))\ z))$$
- Example:
$$(\lambda z.((\lambda f.(f\ x))\ z)) \Rightarrow_\alpha (\lambda z.((\lambda g.(g\ x))\ z))$$

β-reductions

- Let v be a variable and E and E_1 lambda expressions, then
$$((\lambda v.E)\ E_1) \Rightarrow_\beta E[v \to E_1]$$
provided $E[v \to E_1]$ is carried out safely.
- The intuition is that the argument E_1 is passed to the function $(\lambda v.E)$ by substituting E_1 for the formal parameter v.
- An expression of the form
$$((\lambda v.E)\ E_1)$$
is called a β-redex (reduction expression), i.e. an expression that can be β-reduced.

η-reductions

- η-reductions allow us to get rid of extra lambda abstractions.
- For example, we can say
$$(\lambda x.(\mathsf{square}\ x)) \Rightarrow_\eta \mathsf{square}$$
- square is a function that squares its argument, while $(\lambda x.(\mathsf{square}\ x))$ is a function of x that squares x.
- $(\lambda x.(\mathsf{square}\ x))$ reminds us that square takes an argument x, but square is a bit less messy.

η-reductions...

- If E is a lambda expression that denotes a function, and v has no free occurences in E, then
$$(\lambda v.(E\ v)) \Rightarrow_\eta E$$
- The following reduction
$$(\lambda x.(\mathsf{add}\ x\ x)) \Rightarrow_\eta (\mathsf{add}\ x)$$
is invalid since $(\mathsf{add}\ x)$ has a free occurence of x.

Reduction Strategies

Termination

- **Question:**

  Can every lambda expression be reduced to a normal form?
- Consider the expression

$$((\lambda x.(x\ x))\ (\lambda x.(x\ x)))$$

  which reduces to itself:

$$((\lambda \mathtt{x}.(x\ \ x))\ \ (\lambda \mathtt{x}.(\mathtt{x}\ \ \mathtt{x})))\Rightarrow_\beta$$

$$((\lambda x.(x\ \ x))\ \ (\lambda x.(x\ \ x)))\Rightarrow_\beta$$

$$\ldots$$

- **Answer:** No.
- Lambda calculus contains non-terminating reductions.

Paths

- **Question:**

  Is there more than one way to reduce a lambda expression?
- Consider the expression

$$(((\lambda x.(\lambda y.(\mathsf{add}\ y\ ((\lambda z.(\mathsf{mul}\ x\ z))\ 3))))\ 7)\ 5)$$

  Let's try reducing it in two different ways, to see if we'll arrive at different normal forms.
- A lambda expression is in a normal form if we can apply no more β-reductions or δ-reductions.

Paths. . .

$$(((\lambda x.(\lambda y.(\mathsf{add}\ \ y\ \ ((\lambda z.(\mathsf{mul}\ \ x\ \ z))\ \ 3))))\ \ 7)\ \ 5) =$$

$$(((\lambda x.(\lambda y.(\mathsf{add}\ \ y\ \ ((\lambda z.(\mathsf{mul}\ \ x\ \ z))\ \ 3))))\ \ 7)\ \ 5)\Rightarrow_\beta$$

$$((\lambda y.(\mathsf{add}\ \ y\ \ ((\lambda z.(\mathsf{mul}\ \ 7\ \ z))\ \ 3)))\ \ 5)\Rightarrow_\beta$$

$$(\mathsf{add}\ \ 5\ \ ((\lambda z.(\mathsf{mul}\ \ 7\ \ z))\ \ 3))\Rightarrow_\beta$$

$$(\mathsf{add}\ \ 5\ \ (\mathsf{mul}\ \ 7\ \ 3))\Rightarrow_\delta$$

$$(\mathsf{add}\ \ 5\ \ 21)\Rightarrow_\delta$$

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Paths...

$$(((\lambda x.(\lambda y.(\text{add}\ \ y\ \ ((\lambda z.(\text{mul}\ \ x\ \ z))\ \ 3))))\ \ 7)\ \ 5) =$$

$$(((\lambda x.(\lambda y.(\text{add}\ \ y\ \ ((\lambda z.(\text{mul}\ \ x\ \ z))\ \ 3))))\ \ 7)\ \ 5) \Rightarrow_\beta$$

$$(((\lambda x.(\lambda y.(\text{add}\ \ y\ \ (\text{mul}\ \ x\ \ 3))))\ \ 7)\ \ 5) \Rightarrow_\beta$$

$$((\lambda y.(\text{add}\ \ y\ \ (\text{mul}\ \ 7\ \ 3)))\ \ 5) \Rightarrow_\delta$$

$$((\lambda y.(\text{add}\ \ y\ \ 21))\ \ 5) \Rightarrow_\delta$$

$$(\text{add}\ \ 5\ \ 21) \Rightarrow_\delta$$

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Paths

- **Question:**
  Is there more than one way to reduce a lambda expression?
- **Answer:** Yes.

Application Order

- Consider the expression

$$((\lambda y.5)\ ((\lambda x.(x\ x))\ (\lambda x.(x\ x))))$$

- We can either reduce the leftmost redex first:

$$((\lambda y.5)\ ((\lambda x.(x\ x))\ (\lambda x.(x\ x)))) \Rightarrow_\beta 5$$

- or, we can evaluate the rightmost redex every time:

$$((\lambda y.5)\ \ ((\lambda x.(x\ \ x))\ \ (\lambda x.(x\ \ x)))) \Rightarrow_\beta$$

$$((\lambda y.5)\ \ ((\lambda x.(x\ \ x))\ \ (\lambda x.(x\ \ x)))) \Rightarrow_\beta$$

$$((\lambda y.5)\ \ ((\lambda x.(x\ \ x))\ \ (\lambda x.(x\ \ x)))) \Rightarrow_\beta \cdots$$

Application Order...

- The leftmost redex is that redex whose λ is textually to the left of all other redexes within the expression.
- An outermost redex is defined to be a redex which is not contained within any other redex.
- An innermost redex is defined to be a redex which contains no other redex.
- A normal order reduction always reduces the leftmost outermost β-redex (or δ-redex) first.
- A applicative order reduction always reduces the leftmost innermost β-redex (or δ-redex) first.

Finding Redexes

- Remembering that a redex is an expression of the form $((\lambda x.E)\ y)$, find the redexes in this expression:

$$(((\lambda x.(\lambda y.(\text{add}\ \ x\ \ y)))\ \ ((\lambda z.(\text{succ}\ \ z))\ \ 5))\ \ ((\lambda w.(\text{sqr}\ \ w))\ \ 7)) =$$

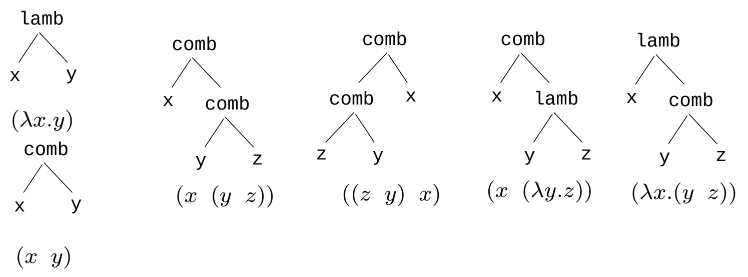
$$(((\lambda x.(\lambda y.(\text{add}\ \ x\ \ y)))\ \ \underbrace{((\lambda z.(\text{succ}\ \ z))\ \ 5)})\ \ ((\lambda w.(\text{sqr}\ \ w))\ \ 7)) =$$

$$(((\lambda x.(\lambda y.(\text{add}\ \ x\ \ y)))\ \ \underbrace{((\lambda z.(\text{succ}\ \ z))\ \ 5)})\ \ \underbrace{((\lambda w.(\text{sqr}\ \ w))\ \ 7)}) =$$

$$(\overbrace{((\lambda x.(\lambda y.(\text{add}\ \ x\ \ y)))\ \ \underbrace{((\lambda z.(\text{succ}\ \ z))\ \ 5)})}\ \ \underbrace{((\lambda w.(\text{sqr}\ \ w))\ \ 7)})$$

Finding Redexes. . .

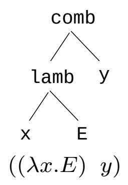
- If we have problems finding the redexes of an expression, we can first draw it as an abstract syntax tree.
- This is a tree that shows the structure of an expression, ignoring syntactic details.



- `comb` stands for an application, `lamb` for an abstraction.

Finding Redexes. . .

- During evaluation we look for a redex, a tree with the following structure:



- I.e., a redex is an abstraction joined with another expression to be passed to the function.

Finding Redexes. . .

- Again, consider the expression:

$$(\overbrace{((\lambda x.(\lambda y.(\text{add}\ x\ y)))\ \underbrace{((\lambda z.(\text{succ}\ z))\ 5)})}\ \underbrace{((\lambda w.(\text{sqr}\ w))\ 7)})$$

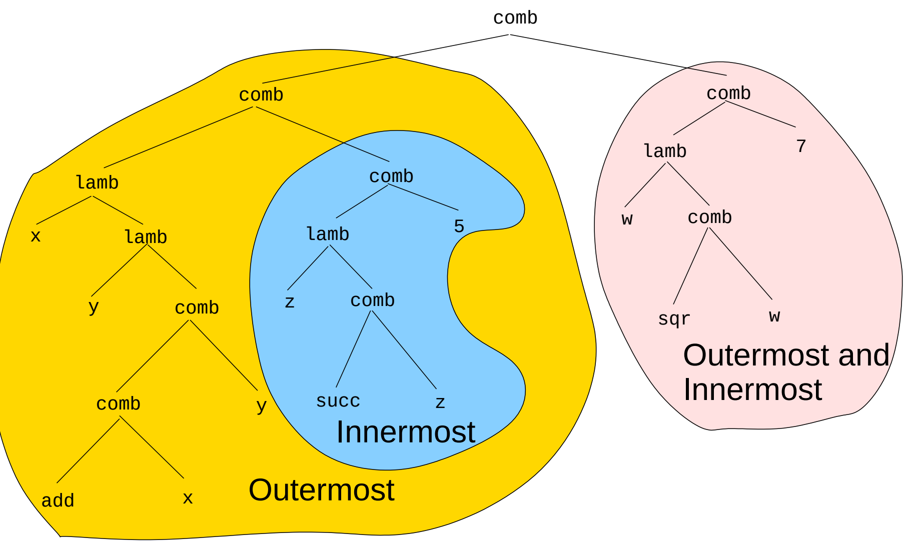
- Its abstract syntax tree is given on the next slide.
- The leftmost outermost redex is

$$(\underbrace{((\lambda x.(\lambda y.(\text{add}\ x\ y)))\ ((\lambda z.(\text{succ}\ z))\ 5))}\ ((\lambda w.(\text{sqr}\ w))\ 7))$$

- The leftmost innermost redex is

$$(((\lambda x.(\lambda y.(\text{add}\ x\ y)))\ \underbrace{((\lambda z.(\text{succ}\ z))\ 5)})\ ((\lambda w.(\text{sqr}\ w))\ 7))$$

Finding Redexes. . .



Church-Rosser Theorem I

- **Question:**

  If there is more than one reduction strategy, does each one lead to the same normal form expression?

- **Theorem:**

  For any lambda expressions E, F and G, if $E \overset{*}{\Rightarrow} F$ and $E \overset{*}{\Rightarrow} G$, there is a lambda expression Z such that $F \overset{*}{\Rightarrow} Z$ and $G \overset{*}{\Rightarrow} Z$.

  - diamond property
  - confluence property

$$\begin{tikzcd}
 & E \arrow[dl, Rightarrow, "*"'] \arrow[dr, Rightarrow, "*"] & \\
F \arrow[dr, Rightarrow, "*"'] & & G \arrow[dl, Rightarrow, "*"] \\
 & Z &
\end{tikzcd}$$

Church-Rosser Theorem I. . .

- **Corollary:**

  For any lambda expressions E, M and N, if $E \overset{*}{\Rightarrow} M$ and $E \overset{*}{\Rightarrow} N$, where M and N are in normal form, M and N are variants of each other (except for changes in variables, using α-reductions).

- **Answer:** Yes, if a lambda expression is in normal form, it is unique, except for changes in bound variables.

Church-Rosser Theorem II

- **Question:**

  Is there a reduction strategy that will guarantee that a normal form expression will be produced, if one exists?

- **Theorem:**

  For any lambda expressions E and N, if $E \overset{*}{\Rightarrow} N$ where N is in normal form, there is a normal order reduction from E to N.

- **Answer:** Yes, normal order reduction will produce a normal form lambda expression, if on exists.

Normal Order Reduction

- A normal-order reduction can have these outcomes:
  1. A unique normal form lambda expression is reached (up to α-reduction).
  2. The reduction never terminates.

Church's Theses

The effectively computable functions on the positive integers are precisely those functions definable in the pure lambda calculus (and computable by Turing Machines).

- It can be shown that every Turing Machine can be simulated by a lambda expression.
- It can be shown that every lambda expression can can be realized by a Turing Machine.
- Hence, Turing Machines and lambda calculus are equivalent.
- Since it's not possible to determine whether a Turing Machine will terminate, it's not possible to determine whether a normal order reduction will terminate.

People

Alonzo Church

From http://www.math.ucla.edu/~hbe/church.pdf

- Born on June 14 (Flag Day), 1903, in Washington, D.C.
- His great-grandfather (also named Alonzo Church) was a professor of mathematics and astronomy.
- His grandfather Alonzo Webster Church was at one time Librarian of the U.S. Senate.
- An airgun incident in high school left Church blind in one eye.
- Church enrolled at Princeton, where his uncles had attended college. He worked part-time in the dining hall to help pay his way.

Alonzo Church. . .

- He was an exceptional student; his fi rst published paper was written while he was an undergraduate.
- He continued graduate work at Princeton, completing a Ph.D. in three years
- While a graduate student, he married Mary Julia Kuczinski, who was training to be a nurse. (This in spite of the fact that his senior class had voted him the most “likely to remain a bachelor”.
- In the summer of 1924, Church stepped o the curb and was hit by a trolley car coming from his blind side; Mary was a nurse-trainee at the hospital.
- Mary was an excellent cook; over the years many a mathematician enjoyed dining at the Church home.
- He and Mary had three children. Alonzo Church, Jr., was born in 1929 , Mary Ann in 1933, and Mildred in 1938. (Mary Ann later married the logician John Addison.)
- Alonzo Church had the polite manners of a gentleman who had grown up in Virginia. He was never known to be rude, even with people with whom he had strong disagreements. A deeply religious person, he was a lifelong member of the Presbyterian church.

Haskell Curry

From http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Curry.html

- Haskell did not show particular interest in mathematics when at high school and when he graduated in 1916 he fully intended to study medicine. He entered Harvard College.
- A major influence on the direction that his studies took was the entry of the United States into World War I in the spring of 1917. Curry wanted to serve his country, and decided that he would be more likely to see action if he had a mathematics training rather than if he continued the pre-medical course he was on.
- The war, however, ended shortly after this (in November) and on 9 December 1918 Curry left the army.

Haskell Curry. . .

- Curry now decided that he would look for a career in electrical engineering and he took a job with the General Electric Company which allowed him to study electrical engineering part-time at the Massachusetts Institute of Technology.
- However he soon discovered that he had a different attitude to the others taking the courses, for he wanted to know why a result was correct when for everyone else it only mattered that it was correct.
- If one imagines that from 1924 when Curry embarked on his doctorate in mathematics at Harvard he at last found the topic for him, then one would be mistaken. He was given a topic in the theory of differential equations by George Birkhoff but he began reading books on logic which seemed to him far more interesting that his research topic.

Readings and References

- Read pp. 145–150, in *Syntax and Semantics of Programming Languages*, by Ken Slonneger and Barry Kurtz, http://www.cs.uiowa.edu/~slonnegr/plf/Book.
- Read pp. 616–618, in Scott.

Acknowledgments

- Much of the material in this lecture on Lambda Calculus is taken from the book *Syntax and Semantics of Programming Languages*, by Ken Slonneger and Barry Kurtz, `http://www.cs.uiowa.edu/~slonnegr/plf/Book`.