Minimum Level Nonplanar Patterns for Trees

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Abstract. We add two minimum level nonplanar (MLNP) patterns for trees to the previous set of tree patterns given by Healy *et al.* [3]. Neither of these patterns match any of the previous patterns. We show that this new set of patterns completely characterize level planar trees.

1 Introduction

Level graphs model hierarchical relationships. A level drawing has all vertices of the same level with the same y-coordinates and all edges strictly y-monotonic. Level planar graphs have level drawings without edge crossings. Hierarchies are special cases in which every vertex is reachable via a y-monotonic path from a source at the top level. Many natural hierarchies occur in the sciences including biological taxonomies, linguistic universal grammars, object-oriented design, multi-tiered social structures, and mathematical hierarchies.

Planar graphs are characterized by forbidden subdivisions of K_5 and $K_{3,3}$ by Kuratowksi's Theorem [7]. The counterpart of this characterization for level planar graphs are the minimum level nonplanar (MLNP) patterns proposed by Healy, Kuusik, and Liepert [3]. A minimal obstructing subgraph with a set of level assignments forcing a crossing constitutes a MLNP pattern.

While Jünger *et al.* provide linear time recognition and embedding algorithms [5, 6] for level planar graphs, swapping the vertices between levels while maintaining planarity can be difficult. Heath and Rosenberg showed that deciding if a planar graph has a proper k-leveling is NP-hard [4]. Finding a matching subgraph of a MLNP pattern can provide a set of candidate vertices to reassign to different levels in order to achieve planarity. Such a method could improve existing hierarchical approaches to drawing directed acyclic graphs (DAGs), such as Sugiyama's algorithm [8] that greedily assigns vertices to levels.

Di Battista and Nardelli [1] provided three level nonplanar patterns for hierarchies (HLNP patterns); cf. Fig. 3. These patterns each consist of three (not necessary) disjoint paths linking a pair of levels that are joined by three pairwise disjoint bridges. If none of the linking paths cross, this condition forces a crossing between one or more bridges. Di Battista *et al.* showed these HLNP patterns were a necessary and sufficient condition for level nonplanar hierarchies. Since these patterns are sufficiently general, they can be extended to determine when level graphs are nonplanar. Healy *et al.* refined these HLNP patterns into a set of MLNP patterns for level graphs. However, the completeness of their characterization was based on the claim that all MLNP patterns must contain a HLNP pattern. This claim does not hold for the counterexample we provide.

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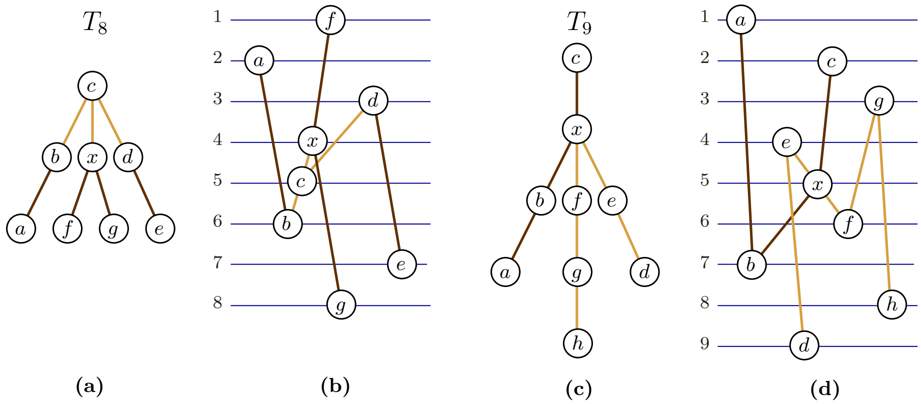


Fig. 1. Labelings preventing the forbidden ULP trees T_8 and T_9 from being level planar.

Estrella *et al.* [2] characterized the set of unlabeled level planar (ULP) trees on n vertices that are level planar over all possible $n!$ labelings of the vertices from 1 to n in terms of the forbidden trees T_8 and T_9 in Fig. 1. The given labelings were used to show that these trees are level nonplanar. Each vertex is assigned to its own level so that its y-coordinate is based on its level. The level nonplanar assignment for T_9 can be shown not to match any of the three HLNP patterns. This forms the basis of our counterexample. For every set of three paths linking any pair of levels in T_9, two of the three linking paths always has a bridge that shares a vertex with the other path. This violates the condition that forces a crossing between the third linking path and the bridge. As a result, this level nonplanar tree does not match any of the MLNP patterns given by Healy *et al.*

Healy *et al.* provides two of the MLNP patterns, P_1 and P_2, for trees that each contain a HLNP pattern; cf. Fig. 2(a) and (b). Both have three disjoint paths linking the top and bottom levels with the three pairwise bridges that form a subdivided $K_{1,3}$. We provide two more MLNP patterns, P_3 and P_4 for level nonplanar trees; cf. Fig. 2(c) and (d) based upon T_9. Both of these patterns consist of two paths that have a common vertex x or subpath $x \rightsquigarrow y$ that lies between two intermediate levels. A crossing is forced between the two paths since x or $x \rightsquigarrow y$ must lie between two different sections of path that they are on in order to avoid a self-crossing of that path.

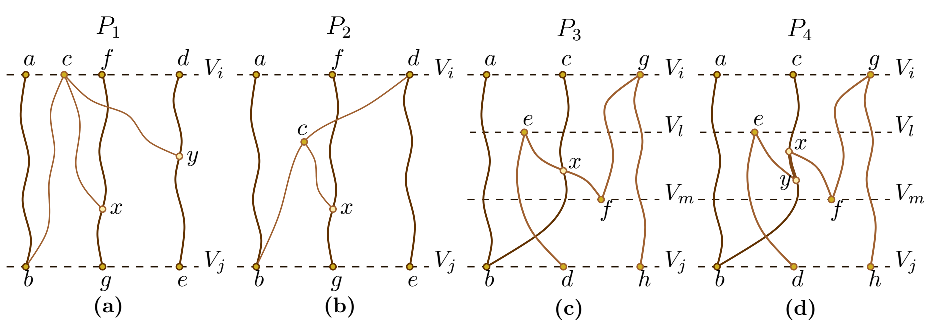


Fig. 2. Four minimum level nonplanar (MLNP) patterns for level nonplanar trees.

2 Preliminaries

A *k-level graph* $G(V, E, \phi)$ on n vertices has *leveling* $\phi : V \to [1..k]$ where every $(u, v) \in E$ either has $\phi(u) < \phi(v)$ if G is directed or $\phi(u) \neq \phi(v)$ if G is undirected. This leveling partitions V into $V_1 \cup V_2 \cup \cdots \cup V_k$ where the *level* $V_j = \phi^{-1}(j)$ and $V_i \cap V_j = \varnothing$ if $i \neq j$. A *proper level graph* only has *short edges* in which $\phi(v) = \phi(u) + 1$ for every $(u, v) \in E$. Edges spanning multiple levels are *long*. A *hierarchy* is a proper level graph in which every vertex $v \in V_j$ for $j > 1$ has at least one incident edge $(u, v) \in E$ to a vertex $u \in V_i$ for some $i < j$.

A *path* p is a nonrepeating ordered sequence of vertices $(v_1, v_2, \ldots, v_t)$ for $t \geq 1$. Let $\text{MIN}(p) = \min\{\phi(v) : v \in p\}$, $\text{MAX}(p) = \max\{\phi(v) : v \in p\}$, and $\mathcal{P}(i, j) = \{p : p \text{ is a path where } i \leq \text{MIN}(p) < \text{MAX}(p) \leq j\}$ are the paths between levels V_i and V_j. A *linking path*, or *link*, $L \in \mathcal{L}(i, j)$ is a path $x \rightsquigarrow y$ in which $i = \text{MIN}(L) = \phi(x)$ and $\text{MAX}(L) = \phi(y) = j$, and $\mathcal{L}(i, j) \subseteq \mathcal{P}(i, j)$ are all paths linking the *extreme levels* V_i and V_j. A *bridge* b is a path $x \rightsquigarrow y$ in $\mathcal{P}(i, j)$ connecting links $L_1, L_2 \in \mathcal{L}(i, j)$ in which $b \cap L_1 = x$ and $b \cap L_2 = y$.

Theorem 1 (Di Battista and Nardelli [1]) *A hierarchy $G(V, E, \phi)$ on k levels is level planar if and only if there does not exist three paths $L_1, L_2, L_3 \in \mathcal{L}(i, j)$ linking levels V_i and V_j for $1 \leq i < j \leq k$ where one of the following hold:*

(a) Links L_1, L_2, and L_3 are completely disjoint and pairwise connected by bridges b_1 from L_1 to L_3, b_2 from L_2 to L_3, and b_3 from L_2 to L_3 such that $b_1, b_2, b_3 \in \mathcal{P}(i, j)$ and $b_1 \cap L_2 = b_2 \cap L_1 = b_3 \cap L_1 = \varnothing$; cf. Fig. 3(a).

(b) Links L_1 and L_2 share a path $C = L_1 \cap L_2 \in \mathcal{P}(i, j)$ starting from endpoint p in V_i or V_j that is disjoint from L_3, $L_1 \cap L_3 = L_2 \cap L_3 = \varnothing$, connected by bridges b_1 from L_1 to L_3 and b_2 from L_1 to L_3 such that $b_1, b_2 \in \mathcal{P}(i, j)$ and $b_1 \cap L_2 = b_2 \cap L_1 = \varnothing$; cf. Fig. 3(b).

(c) Links L_1 and L_2 share a path $C_1 = L_1 \cap L_2 \in \mathcal{P}(i, j)$ starting from endpoint p in V_i and links L_2 and L_3 share a path $C_2 = L_2 \cap L_3 \in \mathcal{P}(i, j)$ starting from endpoint q in V_j such that $C_1 \cap C_2 = \varnothing$. Links L_1 and L_3 are connected by bridge $b \in \mathcal{P}(i, j)$ such that $b \cap L_2 = b \cap C_1 = b \cap C_2 = \varnothing$; cf. Fig. 3(c).

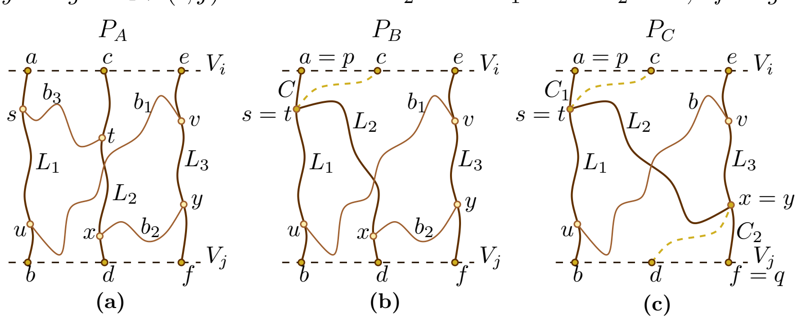


Fig. 3. The three patterns characterizing hierarchies. Patterns P_B and P_C are special cases of P_A. The dashed curves in (b) and (c) are extraneous paths highlighting the relationship P_B and P_C have with P_A. If the bridge b_3 in (a) has no edges, then P_A contains P_B with the extra path $s \rightsquigarrow c$ in (b). Similarly, if both the bridges b_2 and b_3 in (c) have no edges, then P_A contains P_C with the two paths $s \rightsquigarrow c$ and $x \rightsquigarrow d$ in (c).

Any *improper level graph* can be made proper by subdividing all long edges into short edges. A *level drawing* of G has all of its *level-j* vertices in the j^{th} level V_j placed along the *track* $\ell_j = \{(x, k-j) \mid x \in \mathbb{R}\}$, and each edge $(u, v) \in E$ is drawn as a continuous strictly y-monotonic sequence of line segments downwards. A level drawing drawn without edge crossings shows that G is *level planar*. Any level graph can be made into hierarchy by adding a new source with paths to all vertices unreachable via a y-monotonic path to a source. A *pattern* is a set of level nonplanar graphs sharing structural similarities. Each graph matching a *level nonplanar* (LNP) pattern P is level nonplanar. Removing any edge from the underlying graph of a *minimum level nonplanar* (MLNP) pattern gives a level planar graph. A HLNP pattern P is a LNP pattern in which every matching graph is a hierarchy. The previous theorem gave the set of three HLNP patterns.

3 MLNP Patterns for Trees

We begin by providing an extended set of MLNP patterns for trees.

Theorem 2 *A level tree $T(V, E, \phi)$ on k levels is minimum level nonplanar if*

(1) there are three disjoint paths $L_1, L_2, L_3 \in \mathcal{L}(i, j)$ for $1 \leq i < j \leq k$ where P_A of Theorem 1(a) applies and the union of the three bridges $b_1 \cup b_2 \cup b_3$ forms a subdivided $K_{1,3}$ subtree S with vertex c of degree 3 so that either

  (a) c is in V_i and a leaf of S is in V_j as in Fig. 4(a) or c is in V_j and a leaf of S is in V_i, or

  (b) one leaf of S is in V_i and another leaf of S is in V_j as in Fig. 4(b), or

(2) there are four paths $L_1, L_2, L_3, L_4 \in \mathcal{L}(i, j)$ for $1 \leq i < j \leq k$ where L_1 and L_4 are disjoint, L_1 and L_2 join at a vertex in V_j to form a path with endpoints in V_i, L_3 and L_4 join at a vertex in V_i to form a path with endpoints in V_j, and there exist intermediate levels V_l and V_m for some $i < l < m < j$ in which either L_2 or L_3 consists of three subpaths C_1, C_2, and C_3 such that $C_1 \in \mathcal{L}(i, m)$ $\big(d \rightsquigarrow e$ as in Fig. 4(c)$\big)$, $C_2 \in \mathcal{L}(l, m)$ $\big(e \rightsquigarrow f$ as in Fig. 4(c)$\big)$, and $C_3 \in \mathcal{L}(l, j)$ $\big(f \rightsquigarrow g$ as in Fig. 4(c)$\big)$, so that

  (c) $L_2 \cap L_3 = x$ where $l \leq \phi(x) \leq m$ as in Fig. 4(c), or

  (d) $L_2 \cap L_3 = p$ a path $x \rightsquigarrow y$ where $l \leq \phi(x) < \phi(y) \leq m$ as in Fig. 4(d).

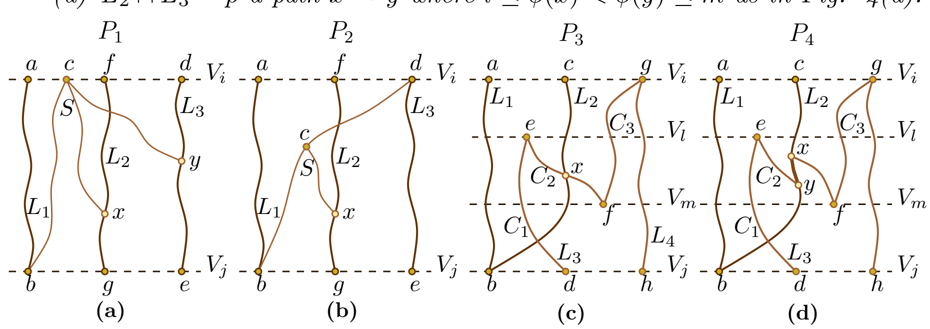


Fig. 4. P_1 of (a) and P_2 of (b) are MLNP patterns $T1$ and $T2$ given by Healy *et al.* [3], respectively. P_3 matches T_9 in Fig. 1. P_4 splits the degree 4 vertex x of P_3 into path $x \rightsquigarrow y$.

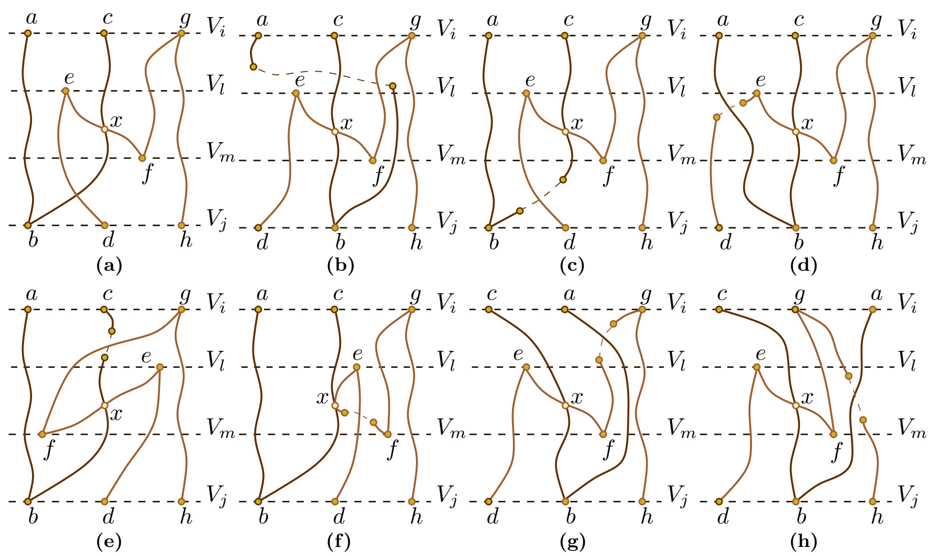


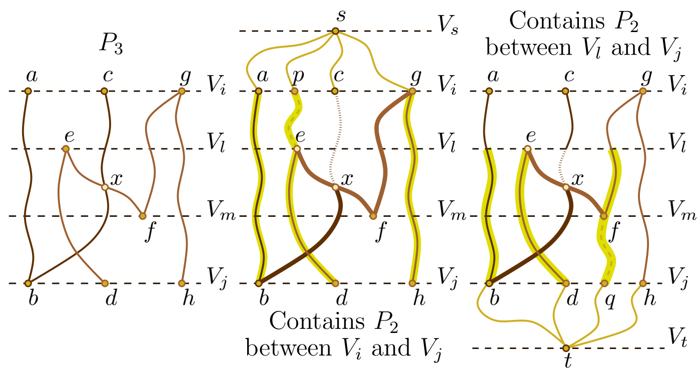
Fig. 5. The various cases of deleting any edge from pattern P_3 in (a). The dashed curves represent the removed edges.

Proof. The description of patterns P_1 and P_2 are more succinctly stated and more closely match notation used in Theorem 1 from [1] than the Healy *et al.* characterization of MLNP T1 and T2 tree patterns given in Section 3.1 of [3]; see the appendix for the original descriptions of T1 and T2.

To show that P_1 and P_2 match the patterns of T1 and T2 is simply a matter of verifying that P_1 and P_2 have the four common conditions listed for T1 and T2 and that the specific conditions for each one are satisfied, all of which is immediate from the definitions of P_1 and P_2. To show that T1 and T2 match P_1 and P_2 requires applying Lemmas 8, 9, 10 in the appendix from [3]. Given that the definitions are equivalent, we apply Theorem 7 in the appendix from [3] to see that P_1 and P_2 are indeed minimum level nonplanar.

We delete an edge from each linking path or bridge of P_3 and P_4 and show how to avoid a crossing in each case.

(i) If an edge is deleted along $a \rightsquigarrow b$ as in Fig. 5(b), then the remaining path can reside under the path $f \rightsquigarrow g \rightsquigarrow h$ where d is then moved left of b.
(ii) If an edge is deleted along $b \rightsquigarrow x$ as in Fig. 5(c), then the other path can take direct advantage of that gap.
(iii) If an edge is deleted along $d \rightsquigarrow e$ as in Fig. 5(d), then $a \rightsquigarrow b$ is drawn through the gap with d left of b.
(iv) If an edge is deleted along $x \rightsquigarrow c$ as in Fig. 5(e), then $d \rightsquigarrow e$ can be drawn right of $b \rightsquigarrow x$ whereas $f \rightsquigarrow g$ is drawn left.
(v) If an edge is deleted along $e \rightsquigarrow f$ as in Fig. 5(f), then $d \rightsquigarrow e$ is drawn left of $b \rightsquigarrow x$ using the gap to avoid a crossing.
(vi) If an edge is deleted along $f \rightsquigarrow g$ as in Fig. 5(g), then $d \rightsquigarrow e$ is drawn left of $b \rightsquigarrow c$ and $a \rightsquigarrow b$ is drawn through the gap of $f \rightsquigarrow g$.
(vii) If an edge is deleted along $g \rightsquigarrow h$ as in Fig. 5(h), then $d \rightsquigarrow e$ is drawn left of $b \rightsquigarrow c$ that is left of $f \rightsquigarrow g$ and $a \rightsquigarrow b$ drawn through the gap of $g \rightsquigarrow h$.



(a) (b) (c)

Fig. 6. P_3 in (a) is augmented from the top in (b) and from the bottom in (c) to form hierarchies with subtrees matching P_2 in both (b) and (c).

The argument used by Estrella *et al.* [2] to show T_9 is level nonplanar easily generalizes for P_3 and P_4. Finally, we observe that in the description of T1 and T2, that both trees have exactly one vertex of degree 3. Since both P_3 has a vertex of degree 4 and P_4 has two vertices of degree 3, neither can match P_1 or P_2. Hence, all four MLNP patterns are distinct. □

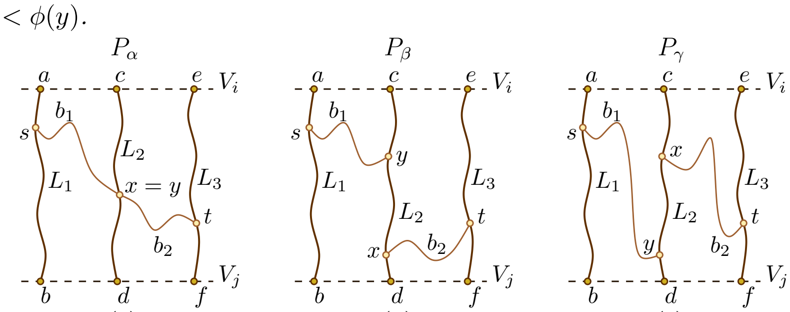
The distinctness of four MLNP patterns shows that P_3 and P_4 are counterexamples to the claim of Theorem 15 of Healy *et al.* [3] that all level nonplanar trees are matched by either T1 or T2. They contended that any level nonplanar graph augmented to form a hierarchy would match the same HLNP pattern *before* being augmented. We next show why this argument fails for P_3.

Lemma 3 *P_3 augmented to form a hierarchy has a subtree matching P_2.*

Proof. Fig. 6 shows the highlighted subtrees that match P_2 when P_3 is augmented to form a hierarchy. However, P_2 does not match P_3 by Theorem 2. □

The next lemma gives the minimal conditions for a MLNP tree pattern.

Lemma 4 *A level nonplanar $T(V, E, \phi)$ on k levels must contain three disjoint links $L_1, L_2, L_3 \in \mathcal{L}(i, j)$ linking levels V_i and V_j for $1 \le i < j \le k$ with bridges b_1 from L_1 to L_2 and b_2 from L_2 to L_3 with $b_1, b_2 \in \mathcal{P}(i, j)$ with endpoints $x = b_1 \cap L_2$ and $y = b_1 \cap L_2$ so that (i) $x = y$, (ii) $\phi(x) > \phi(y)$, or (iii) $\phi(x) < \phi(y)$.*



(a) (b) (c)

Fig. 7. The three minimal patterns that must be part of any MLNP pattern for trees.

Proof. We observe that these conditions fall short of P_A of Theorem 1(a) by only one bridge. By Lemma 10 of [3] in the appendix, P_A is the only HLNP pattern that can match a tree. Hence, so our assertion holds for P_1 and P_2, equivalent to T1 and T2, that Lemmas 8, 9, 10 of [3] show to be special cases of P_A.

Let us assume that we have a MLNP pattern P between levels V_i and V_j and $|i-j|$ is a minimum. Clearly, P must have three (not necessarily) disjoint paths $L_1, L_2, L_3 \in \mathcal{L}(i,j)$ linking levels V_i and V_j. Otherwise, if there were just two linking paths L_1 and L_2, then there can be no path in $\mathcal{P}(i,j)$ joining the two, since otherwise the path would be part of a third linking path. This implies L_1 and L_2 are in separate components contradicting the minimality of P.

At least one of the three paths, say it is L_2, must be joined to the other two paths L_1 or L_3, or P would be disconnected again contradicting the minimality of P. If $b_1 \cap b_2$ form a nonempty path, then $b_1 \cup b_2$ would form a subtree homeomorphic to $K_{1,3}$, and either pattern P_1 or P_2 of Theorem 2 would result. Thus, b_1 and b_2 can share at most one vertex as in P_α of Fig. 7(a). Otherwise there must have endpoints $x = b_1 \cup L_2$ and $y = b_2 \cup L_2$ where either $\phi(x) > \phi(y)$ as in P_β of Fig. 7(b) or (iii) $\phi(x) < \phi(y)$ as in P_γ of Fig. 7(c). We observe that P_α matches P_3 and P_γ matches P_4. □

We next show that P_4 is easily derived from P_3.

Lemma 5 *P_4 is only the distinct MLNP pattern for trees that be formed from P_3 (by splitting the degree-4 vertex) not containing a subtree matching P_2.*

Proof. Fig. 8 shows the three ways in which the degree-4 vertex of P_3 can be split into two degree-3 vertices. Two of the ways contain subtrees that match P_2 for intermediate levels.

Applying definition of P_3 given in Theorem 2, the links L_2 and L_3 share a common vertex x as in Fig. 8(a). If x is replaced by a path $x \rightsquigarrow y$, then there are three cases: (i) $L_2 \cap L_3 = \varnothing$, (ii) $L_2 \cap L_3 = x \rightsquigarrow y$ with $\phi(x) > \phi(y)$, and (iii) $L_2 \cap L_3 = x \rightsquigarrow y$ with $\phi(x) < \phi(y)$. For (i) and (ii), P_2 matches a subtree between levels V_l and V_m as in Fig. 8(b) and between levels V_i and V_j as in Fig. 8(c). The final case (iii), which is P_4. □

We conclude by showing the completeness of our characterization for level nonplanar trees.

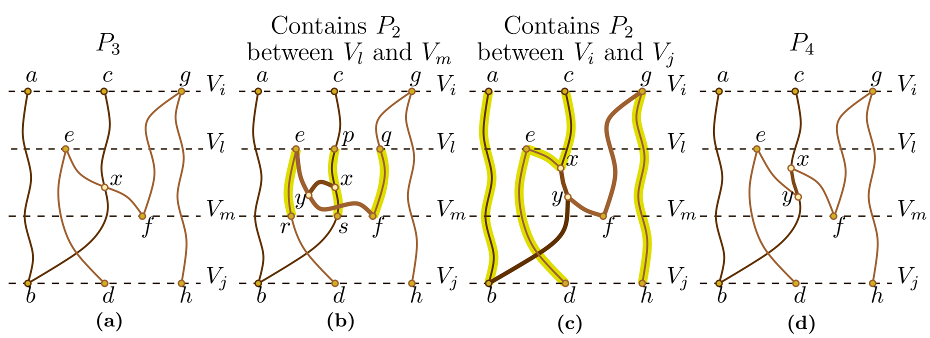


Fig. 8. The three ways in which the degree-4 vertex of P_3 can be split into two vertices of degree 3, the last of which yields P_4. The other two match P_2.

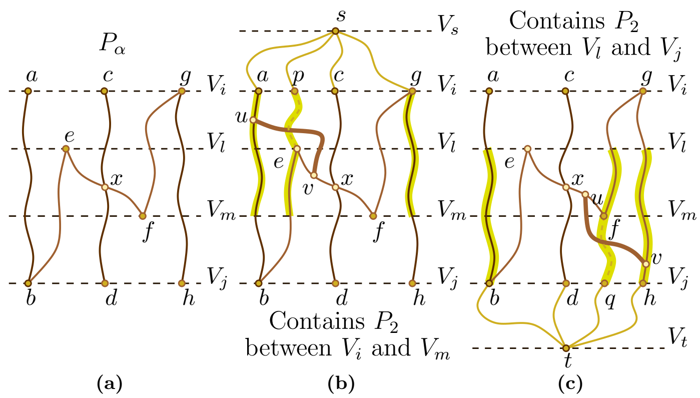


Fig. 9. Examples of pattern P_α in (a) being augmented to form a hierarchy in (b) and (c).

Theorem 6 *A level tree T is level nonplanar if and only if T has a subtree matching one of the minimum level nonplanar patterns P_1, P_2, P_3, or P_4.*

Proof Sketch: We note as in the case of the proof of Lemma 3 in which P_3 is augmented to form a hierarchy, one of the HLNP patterns must apply once the pattern has been augmented. Since this augmentation can always be done to avoid introducing a cycle between levels V_i and V_j, either pattern P_1 or P_2 must match a subtree of the augmented pattern by Lemma 10 of [3].

Assume there is a MLNP tree pattern P that matches P_α of Lemma 4 that does not match P_1 or P_2. There are several cases to consider how the bridges of P_α in P could spans levels between V_i and V_j. For each case we augment P to form a hierarchy. We only give the simplest case to illustrate how either P must match P_1 or P_2 or contain a cycle preventing it from matching a tree. All the other cases are similar variants.

Suppose that neither bridge of the P_α in P is strictly y-monotonic. Then P has a bend at e in level V_l in one bridge and a bend at f in level V_m in the other as in Fig. 9(a) for some $i < l < m < j$. Each bend would require being augmented with a path from the source when forming a hierarchy from above or below as was the case of P_3 being augmented in Fig. 6.

We augment P with a path $p \rightsquigarrow e$ from V_i to V_l to form P', a hierarchy, that must match P_1 or P_2. We observe between levels V_i and V_m, we have four linking paths. A third bridge $u \rightsquigarrow v$ must be present in P' that is part of a subtree S homeomorphic to $K_{1,3}$. Fig. 9(b) gives one such example. While P' matches P_2 between levels V_i and V_m, we see that between levels V_i and V_j, P must have had the cycle $u \rightsquigarrow v \rightsquigarrow e \rightsquigarrow b \rightsquigarrow u$, contradicting P being a tree pattern. By inspection, any other placement of $u \rightsquigarrow v$ to connect three of the four linking paths to form P_1 or P_2 similarly implies a cycle in P.

Hence, P cannot contain any more edges than those of P_α without matching P_1 or P_2. We observe that P_α consists of two paths sharing a common vertex x. Given the minimality of P in minimizing $|i-j|$, one path has both endpoints in V_i with one one vertex in V_j that can be split into linking paths $L_1, L_1 \in \mathcal{L}(i,j)$. Similarly, the other has both endpoints in V_j with one vertex in V_i that can also be split into the linking paths $L_3, L_4 \in \mathcal{L}(i,j)$. In P_3 of Fig. 9(a), L_1 is $a \rightsquigarrow b$, L_2 is $b \rightsquigarrow d \rightsquigarrow x \rightsquigarrow c$, L_3 is $d \rightsquigarrow x \rightsquigarrow f \rightsquigarrow g$, and L_4 is $g \rightsquigarrow h$.

For P to be level nonplanar, a crossing must be forced between these two paths. This can be accomplished by having L_2 or L_3 meet the condition of P_3 of three subpaths $C_1 \in \mathcal{L}(i,m)$ linking V_i to V_m, $C_2 \in \mathcal{L}(l,m)$ linking V_l to V_m, and $C_3 \in \mathcal{L}(l,j)$ linking V_l to V_j. This is not the case for the P_α in Fig. 9(a) since the $x \rightsquigarrow c$ portion of L_2 does not reach level V_m, and the $x \rightsquigarrow d$ portion of L_3 does not reach level V_l. So for P not to match P_3, at least one subpath of both L_2 and L_3 from x to V_i or V_j must strictly monotonic as was the case in Fig. 9(a). However, in this case P can always be drawn without crossings. This leaves P_3 as the only possibility of a MLNP pattern matching P_α that does not P_1 or P_2. □

4 Conclusion and Future Work

The sufficiency argument of the MLNP patterns used by Healy *et al.* is flawed in its contention that all MLNP patterns contain a HLNP pattern. Given this flaw, there remains the very likely possibility of the characterization of Healy *et al.* omitting some MLNP patterns with cycles.

We provided two new MLNP patterns for trees and showed that the new set of four was sufficient. We presented a new approach for showing sufficiency based upon pattern augmentation to form HLNP patterns. However, our approach heavily relied on the underlying graph of the pattern forming a tree and avoiding cycles. For future work remains the open problem of finding the remaining set, if any, of MLNP patterns for graphs with cycles and proving they are sufficient to complete the characterization for all level planar graphs.

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Appendix

Characterization of patterns T1 and T2 from Healy *et al.* in Section 3.1 of [3]:

"Let i and j be the extreme levels of a pattern and let x denote a root vertex with degree 3 that is located on one of the levels $i, \ldots, j$. From the root vertex emerge 3 subtrees that have the following common properties (see Fig. 2 for illustrations of two typical patterns):

- each subtree has at least one vertex on both extreme levels;
- a subtree is either a chain or it has two branches which are chains;
- all the leaf vertices of the subtrees are located on the extreme levels, and if there is a leaf vertex v of a subtree S on an extreme level $l \in \{i, j\}$ then v is the only vertex of S on the extreme level l;
- those subtrees which are chains have one or more non-leaf vertices on the extreme level opposite to the level of their leaf vertices.

The location of the root vertex distinguishes the two characterizations.

(T1) The root vertex x is on an extreme level $l \in \{i, j\}$ (see Fig. 2(a)):

- at least one of the subtrees is a chain starting from x, going to the opposite extreme level of x and finishing on x's level;

(T2) The root vertex x is on one of the intermediate levels $l, i < l < j$ (see Fig. 2(b)):

- at least one of the subtrees is a chain that starts from the root vertex, goes to the extreme level i and finishes on the extreme level j;
  at least one of the subtrees is a chain that starts from the root vertex, goes to the extreme level j and finishes on the extreme level i."

Note that Fig. 2(a) and (b) of [3] correspond to our Figs. 2(a) and (b).

Next we state Theorem 2 and Lemmas 3, 4, and 5 of [3] with slight rewording to match our own terminology and previous theorems.

Theorem 7 (Healy *et al.* Theorem 2) *A subgraph matching either of the two tree characterizations $T1$ or $T2$ is MLNP.*

Lemma 8 (Healy *et al.* Lemma 3) *If HLNP pattern P_A of Theorem 1(a) matches a tree then each one of the paths L_1, L_2, L_3 contains only one vertex being the end vertex of a bridge.*

Lemma 9 (Healy *et al.* Lemma 4) *If HLNP pattern P_A of Theorem 1(a) matches a tree then its bridges must form a subgraph homeomorphic to $K_{1,3}$.*

Lemma 10 (Healy *et al.* Lemma 5) *The only HLNP pattern that can be matched to a tree is P_A of Theorem 1.*