Constrained Simultaneous and Near-Simultaneous Embeddings

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Abstract. A *geometric simultaneous embedding* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with a bijective mapping of their vertex sets $\gamma : V_1 \to V_2$ is a pair of planar straight-line drawings Γ_1 of G_1 and Γ_2 of G_2, such that each vertex $v_2 = \gamma(v_1)$ is mapped in Γ_2 to the same point where v_1 is mapped in Γ_1, where $v_1 \in V_1$ and $v_2 \in V_2$.
In this paper we examine several constrained versions of the geometric simultaneous embedding problem as well as a more relaxed version in which instead of exactly simultaneous we look for near-simultaneous embeddings. We show that if the input graphs are assumed to share no common edges this does not seem to yield large classes of graphs that can be simultaneously embedded. Further, we show that if a prescribed combinatorial embedding for each input graph must be preserved, then we can answer some of the problems that are still open for geometric simultaneous embedding. Finally, we present some positive and negative results on the near-simultaneous embedding problem, in which vertices are not forced to be placed exactly in the same, but just in "near" points in different drawings.

1 Introduction

Graph drawing techniques are commonly used to visualize relationships between objects, where the objects are the vertices of the graph and the relationships are captured by the edges in the graph. Simultaneous embedding is a problem that arises when visualizing two or more relationships defined on the same set of objects. If the graphs that correspond to these relationships are planar, the aim of simultaneous embedding is to find point locations in the plane for the vertices of the graphs, so that each of the graphs can be realized on the same point-set without edge crossings. To ensure good readability of the drawings, it is preferable if the edges are drawn as straight-line segments. This problem is known as *geometric simultaneous embedding*. It has been shown that only a few classes of graphs can be embedded simultaneously with straight-line segments. In particular, Brass *et al.* [1], Erten and Kobourov [6], and Geyer *et al.* [11] showed that three paths, a planar graph and a path, and two trees do not admit geometric simultaneous embeddings. On the positive side, an algorithm for geometric simultaneous embedding of two caterpillars [1] is the strongest known result.

As geometric simultaneous embedding turns out to be very restrictive, it is natural to relax some of the constraints of the problem. Not insisting on straight-line edges led to positive results such as a linear time algorithm by Erten and Kobourov for embedding any pair of planar graphs with at most three bends per edge, or any pair of trees with at most two bends per edge [6]. In such results it is allowed for an edge connecting a pair of vertices to be represented by different Jordan curves in different drawings, something not possible when edges are straight-line segments. As this can be detrimental to the readability of the drawings, several papers considered a slightly more constrained version of this problem, namely, *simultaneous embedding with fixed edges*. In this version of the problem bends are allowed, however, an edge connecting the same pair of vertices must be drawn in exactly the same way in all drawings. Di Giacomo and Liotta [4] showed that outerplanar graphs can be simultaneously embedded with fixed edges with paths or cycles using at most one bend per edge. Frati [9] showed that a planar graph and a tree can also be simultaneously embedded with fixed edges.

The problem of simultaneous graph embedding is related to the problem of computing graph thickness. In particular, by showing that unions of disjoint cycles have a geometric simultaneous embedding, Duncan *et al.* [5] proved that graphs of maximum degree four have geometric thickness two. Using a similar relationship between simultaneous embedding with fixed edges and the weak realizability problem, Gassner *et al.* [10] showed that testing whether three graphs have simultaneous embedding with fixed edges is an NP-Complete problem. In a slightly different setting, Erten and Kobourov [7] showed how to simultaneously embed a planar graph and its dual.

Studying the existing variants of simultaneous embedding has led to practical embedding algorithms for some classes of graphs and techniques for simultaneous embedding have been used in visualizing evolving and dynamic graphs [2]. However, from a theoretical point of view, many problems remain open, while in practice algorithms that attempt to apply these ideas to evolving and dynamic graphs do not provide any guarantees on the quality of the resulting layouts. With this in mind, we consider three further variants of the geometric simultaneous embedding problem.

Graph Classes	Geometric	No Shared	Fixed Embedding	No Shared, Fixed Embedding
path + path	YES [1]	YES [1]	YES [1]	YES [1]
star + path	YES [1]	YES [1]	YES *Sec. 4.1*	YES *Sec. 4.1*
double-star + path	YES [1]	YES [1]	?	YES *Sec. 4.1*
caterpillar + path	YES [1]	YES [1]	?	?
caterpillar + caterpillar	YES [1]	YES [1]	NO *Sec. 4.2*	NO *Sec. 4.2*
3 **paths**	NO [1]	?	NO [1]	?
tree + path	?	?	?	?
tree + cycle	?	?	?	?
tree + caterpillar	?	?	NO *Sec. 4.2*	NO *Sec. 4.2*
outerplanar + path	?	?	NO *Sec. 4.3*	NO *Sec. 4.3*
outerplanar + caterpillar	?	?	NO *Sec. 4.2*	NO *Sec. 4.2*
outerplanar + cycle	?	?	NO *Sec. 4.3*	NO *Sec. 4.3*
tree + tree	NO [11]	?	NO [11]	NO *Sec. 4.2*
outerplanar + tree	NO [11]	?	NO [11]	NO *Sec. 4.2*
outerplanar + outerplanar	NO [1]	?	NO [1]	NO *Sec. 4.2*
planar + path	NO [6]	NO *Sec. 3*	NO [6]	NO *Sec. 3*
planar + tree	NO [6]	NO *Sec. 3*	NO [6]	NO *Sec. 3*
planar + planar	NO [6]	NO *Sec. 3*	NO [6]	NO *Sec. 3*

Fig. 1. A summary of the known results and contributions of this paper. In particular, we survey results in geometric simultaneous embedding (Geometric), geometric simultaneous embedding assuming the graphs do not share common edges (No Shared), geometric simultaneous drawing with fixed embedding (Fixed Embedding), geometric simultaneous drawing with fixed embedding and no common edges (No Shared, Fixed Embedding).

Most of the proofs about the non-existence of simultaneous embeddings exploit the presence of common edges between the graphs that have to be drawn. Hence, it is natural to ask whether larger classes of graphs have geometric simultaneous embeddings if no edges are shared by input graphs. In Section 3 we answer in the negative for planar graph-path pairs, generalizing the result in [6], where it is shown that a planar graph and a path that share edges do not admit a geometric simultaneous embedding.

In Section 4 we consider the problem of geometric simultaneous embedding where the individual embeddings for the input graphs are fixed. We call this setting *geometric simultaneous embedding with fixed embeddings*. This is a more restrictive variant than geometric simultaneous embedding and therefore the negative results for geometric simultaneous embedding remain valid here. We show that some classes of graphs that have geometric simultaneous embeddings do not admit one with individually fixed embeddings. In particular, we prove such a negative result for caterpillar-caterpillar pairs. Moreover, in the fixed embedding setting we are able to solve problems that are still open for geometric simultaneous embedding. Namely, we provide an outerplanar-path pair that has no geometric simultaneous drawing with fixed embedding. All the negative results claimed are still valid if the input graphs are assumed to not share edges. In the aim of establishing which classes of graphs admit geometric simultaneous drawings with fixed embedding, we also partially cover the known positive results for geometric simultaneous embedding, by showing that a star and a path can always be realized and that a double-star and a path can always be realized if they do not share edges.

In the quest for more practical setting where we can still guarantee some properties of the resulting embeddings, we study a variant we call *geometric near-simultaneous embedding*. In this setting (Section 5), edges are drawn as straight-line segments but vertices that represent the same entity in different input graphs can be placed not exactly in the same point but in points that are just near each other. We show that even this version is restrictive. Namely, assuming that vertices are placed on the integer grid, we show that there exist pairs of n-vertex planar graphs in which vertices that represent the same entity in different graphs must be placed in points that are at distance $\Omega(n)$. We finally consider input graphs that are "similar" in their combinatorial structure, and we describe algorithms which guarantee that vertices representing the same entity are displaced only by a constant distance from one drawing to the next. Such algorithms can be used to guarantee limited displacement in dynamic graph drawings.

2 Preliminaries

Here we summarize some of the basic terminology used in this paper; further graph drawing definitions can be found in the surveys by Di Battista *et al.* [3] and by Kaufmann and Wagner [14].

A *straight-line drawing* of a graph is a mapping of each vertex to a unique point in the plane and of each edge to a segment between the endpoints of the edge. A *planar drawing* is one in which no two edges intersect. A *planar graph* is a graph that admits a planar drawing. It is a well-known result [8] that every planar graph admits a planar straight-line drawing. A *grid* drawing is one in which every vertex is placed at a point with integer coordinates in the plane. An *embedding* of a graph is a circular ordering of the edges incident on each vertex of G. An embedding of a graph specifies the faces in any drawing respecting such an embedding, even though the embedding does not determine which one is the *external face*. A graph is *triconnected* if for every pair of distinct vertices there exist three vertex-disjoint paths connecting them. A triconnected graph has an unique embedding, up to a reversal of its adjacency lists.

An *outerplanar graph* is a graph that admits a drawing in which all the vertices are incident to the same face. The embedding of the outerplanar graph in an outerplanar drawing is called an *outerplanar embedding*. *Trees* are connected acyclic graphs and they are a subclass of the outerplanar graphs. The *degree* of a vertex is the number of its neighbors. A *leaf* is a vertex of a tree with degree 1. A *path* is a tree in which every vertex, other than the leaves, has degree 2. A *caterpillar* is a tree in which the removal of all the leaves and their incident edges yields a path. A *star* (*double-star*) is a caterpillar with only one vertex (two vertices) of degree greater than one.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two n-vertex planar graphs with a bijective mapping $\gamma : V_1 \rightarrow V_2$ between their vertices. A *geometric simultaneous embedding* of two graphs exists if a pair of straight-line drawings Γ_1 and Γ_2 of G_1 and of G_2 can be found, such that: (i) each of Γ_1 and Γ_2 is planar, and (ii) each vertex $v_2 = \gamma(v_1)$, with $v_1 \in V_1$ and $v_2 \in V_2$, is mapped in Γ_2 to the same point where v_1 is mapped in Γ_1.

3 Simultaneous Embedding without Common Edges

In this section we consider the geometric simultaneous embedding of graphs that do not share common edges. We show that a planar graph and a path cannot be drawn simultaneously even if they do not share common edges, thus extending an earlier result [6] for a planar graph and a path that do share edges. We revisit the problem of embedding simultaneously graphs not sharing edges in the conclusions (Section 6).

Let G^* be the triconnected planar graph on nine vertices v_1, v_2, ..., v_9 shown in Fig. 2(a). Since G^* is triconnected, it has the same faces in each of its planar embeddings. Let F^* denote the triangular face $\Delta v_1 v_3 v_9$ and P^* be the path $(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)$.

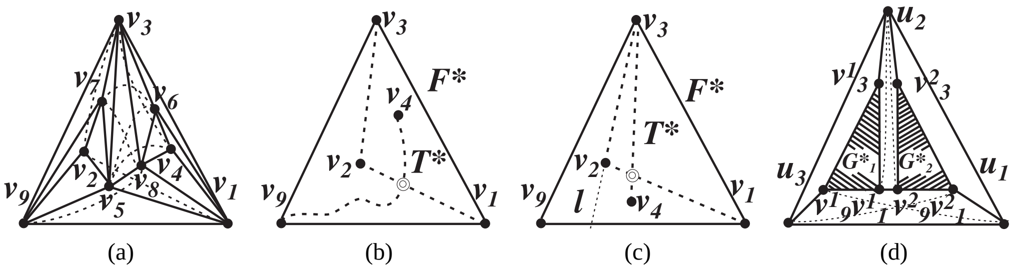


Fig. 2. (a) Triconnected planar graph G^* drawn with solid edges and path P^* drawn with dashed edges; (b) Embedding vertex v_4 inside T^* creates a crossing between the subpath of P^* connecting v_1 and v_3 and the subpath of P^* connecting v_4 and v_9; (c) Embedding vertex v_4 outside T^* creates a crossing between edges (v_1, v_2) and (v_3, v_4) of P^*; (d) Triconnected planar graph G drawn with solid edges and path P drawn with dashed edges.

Lemma 1. *There does not exist a geometric simultaneous embedding of G^* and P^* in which the external face of G^* is F^*.*

Proof: Note that all vertices of G^*, other than v_1, v_3 and v_9, are contained inside F^* as F^* is the external face of G^*. Consider the triangle T^* formed by the edges (v_1, v_2), (v_2, v_3) of P^*, and by the edge (v_1, v_3) of G^*. Since vertex v_9 is incident to F^*, it must lie outside T^*. Let l be the line passing through v_2 and v_3; l separates the plane in two open half-planes, one containing v_9, called the *exterior part* of l, and one not containing v_9, called the *interior part* of l. We show that every placement of v_4 leads to a crossing in the drawing of the path if the planarity of the drawing of G^* is preserved. If v_4 is placed inside T^* then the subpath of P^* composed of the edges (v_1, v_2) and (v_2, v_3) crosses the subpath of P^* connecting v_4, that lies inside T^*, and v_9, that lies outside T^*; see Fig. 2(b). Suppose v_4 is placed outside T^*. Since vertex v_4 (vertex v_2) must lie inside triangle $\Delta v_1 v_3 v_5$ (inside triangle $\Delta v_3 v_5 v_9$), the clockwise order of the edges $(v_3, v_1), (v_3, v_5), (v_3, v_9)$ of G^* and the edges $(v_3, v_4), (v_3, v_2)$ of P^* around v_3 must be $(v_3, v_1), (v_3, v_4), (v_3, v_5), (v_3, v_2), (v_3, v_9)$. Therefore v_4 is in the *interior part* of l and hence edge (v_1, v_2) crosses edge (v_3, v_4) in P^*; see Fig. 2(c). □

Theorem 1. *There exist a planar graph G, a path P, and a mapping between their vertices such that: (i) G and P do not share edges, and (ii) G and P have no geometric simultaneous embedding.*

Proof: We will construct graph G and path P out of two copies of G^* and P^* described above. In particular, let G_1^* and G_2^* be two copies of the planar graph G^*. G_1^* and G_2^* have nine vertices each, and we denote by v_i^j the vertex of G_j^* that corresponds to the vertex v_i in G^*, where $j = 1, 2$ and $i = 1, \ldots, 9$.

Let G be the graph composed of G_1^* and G_2^* together with three additional vertices u_1, u_2, and u_3 and eight additional edges (u_1, u_2), (u_1, u_3), (u_2, u_3), (u_1, v_1^2), (u_2, v_3^1), (u_2, v_3^2), (u_3, v_9^1), and (v_1^1, v_9^2); see Fig. 2(d). It is easy to see that G is a triconnected planar graph. Therefore G has exactly one planar embedding and it has the same faces in each of its plane drawings.

Let F_1^* and F_2^* denote the cycles (v_1^1, v_3^1, v_9^1) and (v_1^2, v_3^2, v_9^2); note that these cycles are faces of G_1^* and G_2^*. Let P be the path $(u_1, v_9^1, v_8^1, v_7^1, v_6^1, v_5^1, v_4^1, v_3^1, v_2^1, v_1^1, u_2, v_9^2, v_8^2, v_7^2, v_6^2, v_5^2, v_4^2, v_3^2, v_2^2, v_1^2, u_3)$. It is easy to see that G and P do not share edges. Note that the subpaths of P induced by the vertices of G_1^* and by the vertices of G_2^* play the same role that path P^* plays for graph G^* in Lemma 1.

We now show that every plane drawing Γ of G determines a non-planar drawing of P. Consider the particular embedding $\mathcal{E}_G$ of G obtained by choosing $\Delta u_1 u_2 u_3$ as external face; see Fig. 2(d). Note that choosing any face internal to F_1^* (F_2^*) in $\mathcal{E}_G$ as external face of Γ leaves G_2^* (G_1^*) embedded with external face F_2^* (F_1^*) and that choosing any face external to both F_1^* and F_2^* in $\mathcal{E}_G$ as the external face of Γ leaves G_1^* and G_2^* embedded with external face F_1^* or F_2^*, respectively. Hence, we can apply Lemma 1 and conclude that there does not exist a simultaneous embedding of G and P. □

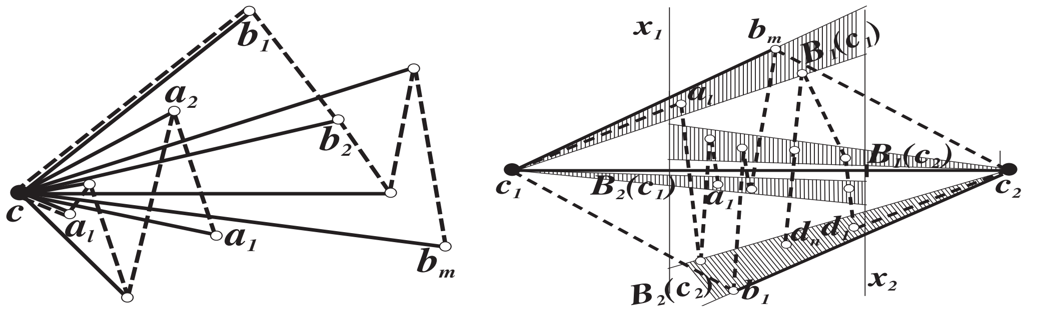
4 Simultaneous Drawing with Fixed Embedding

Next, we examine the possibility of embedding graphs simultaneously with straight-line edges and with fixed embeddings for the input graphs. In this setting we show that star-path pairs and double-star-path pairs not sharing edges can be drawn simultaneously, while there are caterpillar-caterpillar pairs that cannot. We also give an outerplanar graph and a path that cannot be drawn simultaneously given a fixed embedding.

4.1 Simultaneous Drawing of Stars, Double-Stars and Paths with Fixed Embedding

Let P be an n-vertex path and let S be an n-vertex star with fixed embedding $\mathcal{E}$ and *center* c. Note that S and P share at least one and at most two edges. Let $P = (a_1, a_2, \ldots, a_l, c, b_1, b_2, \ldots, b_m)$, where one among the sequences $(a_1, a_2, \ldots, a_l)$ and $(b_1, b_2, \ldots, b_m)$ could be empty. Draw S with c as the leftmost point and all of the edges in an order around c consistent with $\mathcal{E}$ and so that edge (c, b_1), if it exists, is the *uppermost* edge of S. It is easy to ensure that the x-coordinate of a vertex b_i is greater than the x-coordinate of a vertex a_j, with $1 \leq i \leq m$ and $1 \leq j \leq l$, that the x-coordinate of a vertex b_i is greater than the x-coordinate of a vertex b_j, with $1 \leq j < i \leq m$, and that the x-coordinate of a vertex a_i is greater than the x-coordinate of a vertex a_j, with $1 \leq i < j \leq l$; see Fig. 3(a). The resulting drawing of S is clearly planar. It is also easy to see that the path P is not self intersecting as it is realized by two x-monotone curves joined by an edge that is higher than every other edge of P. This yields the following result:

Theorem 2. *An n-vertex star and an n-vertex path admit a geometric simultaneous embedding in which the star has a fixed prescribed embedding.*



(a) (b)

Fig. 3. (a) Simultaneous embedding of a star and a path; (b) Simultaneous embedding of a double-star and a path not sharing edges.

Now let P be an n-vertex path and let D be an n-vertex double-star with a prescribed embedding $\mathcal{E}$ and with *centers* c_1 and c_2. Suppose that D and P do not share edges. Let $P = (a_1, a_2, \dots, a_l, c_1, b_1, b_2, \dots, b_m, c_2, d_1, d_2, \dots, d_n)$. Note that the sequences $(a_1, a_2, \dots, a_l)$ and $(d_1, d_2, \dots, d_n)$ could be empty, while $m \geq 2$. Observe also that b_1 is a neighbor of c_2 and b_m is a neighbor of c_1 in D; see Fig. 3(b). The edges incident on c_1 (incident on c_2), except for (c_1, c_2), are grouped into two *bundles* $B_1(c_1)$ and $B_2(c_1)$ (resp. $B_1(c_2)$ and $B_2(c_2)$). $B_1(c_1)$ is made up of the edges starting from (c_1, b_m) until, but not including, (c_1, c_2) in the clockwise order of the edges incident on c_1. $B_2(c_1)$ is made up of the edges starting from (c_1, c_2) until, but not including, (c_1, b_m) in the clockwise order of the edges incident on c_1. The other two bundles $B_1(c_2)$ and $B_2(c_2)$ are defined analogously. P is divided into three subpaths: a subpath $P_1 = (c_1, a_l, a_{l-1}, \dots, a_2, a_1)$, a subpath $P_2 = (c_1, b_1, b_2, \dots, b_m, c_2)$, and a subpath $P_3 = (c_2, d_1, d_2, \dots, d_n)$.

Draw (c_1, c_2) as an horizontal line segment, with c_1 on the left. $B_1(c_1)$ and $B_2(c_1)$ ($B_1(c_2)$ and $B_2(c_2)$) are drawn inside *wedges* centered at c_1 (resp. centered at c_2) and directed rightward (resp. directed leftward), with $B_1(c_1)$ above (c_1, c_2) and $B_2(c_1)$ below (c_1, c_2) (resp. with $B_1(c_2)$ above (c_2, c_1) and $B_2(c_2)$ below (c_2, c_1)). Such wedges are disjoint and have the further property that there exists an interval $[x_1, x_2]$ of the x-axis that is common to all the wedges. $[x_1, x_2]$ is a sub-interval of the x-extension of the edge (c_1, c_2). Draw each edge inside the wedge of its bundle, respecting $\mathcal{E}$ and so that the following rules are observed: the x-coordinate of a vertex b_i is greater than the x-coordinate of a vertex a_j, with $1 \leq i \leq m$ and $1 \leq j \leq l$; the x-coordinate of a vertex d_k is greater than the x-coordinate of a vertex b_i, with $1 \leq k \leq n$ and $1 \leq i \leq m$; the vertices of P_1 have increasing x-coordinates; the vertices of P_2 have increasing x-coordinates; and the vertices of P_3 have decreasing x-coordinates. Each vertex has an x-coordinate in the open interval (x_1, x_2). Edge (c_1, b_m) (edge (c_2, b_1)) of D is drawn so high (resp. so low) that edge (c_2, b_m) (resp (c_1, b_1)) of P does not create crossings with the other edges of the path. The absence of crossings in the drawing of D follows from the fact that its edges are drawn inside disjoint regions of the plane. The absence of crossings in the drawing of P follows from (1) the absence of crossings in the drawings of its subpaths, which in turn follows from the strictly increasing or decreasing x-coordinate of its vertices; and (2) from the fact that the subpaths occupy disjoint regions, except for edges (c_1, b_1) and (c_2, b_m) which do not create crossings, as already discussed. Thus, we have the following result:

Theorem 3. *An n-vertex double-star and an n-vertex path not sharing edges admit a geometric simultaneous embedding in which the double-star has a fixed prescribed embedding.*

4.2 Simultaneous Drawing of Two Caterpillars with Fixed Embedding

Insisting on a fixed embedding when simultaneously embedding planar graphs is a very restrictive requirement as shown by the following theorem:

Theorem 4. *It is not always possible to find a geometric simultaneous embedding for two caterpillars with fixed embeddings.*

Proof: Let C_1 and C_2 be the two caterpillars with fixed embeddings $\mathcal{E}_1$ and $\mathcal{E}_2$ and a bijective mapping $\gamma(x) = x$ between their vertices; see Fig. 4(a-b). We now show that there does not exist a geometric simultaneous embedding of C_1 and C_2 in which C_1 and C_2 respect $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively.

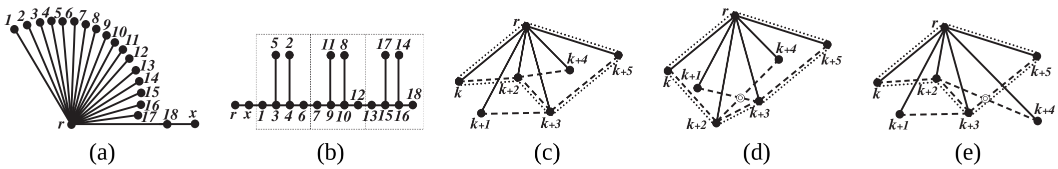


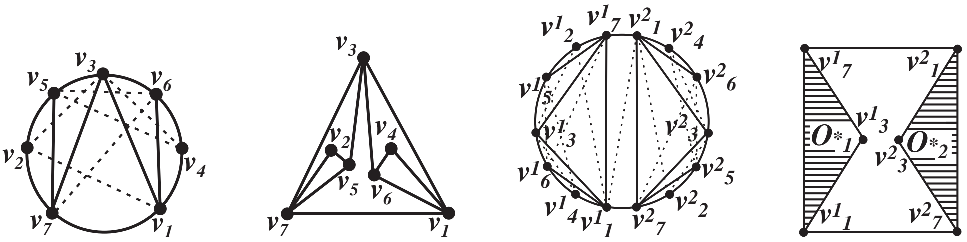
Fig. 4. (a) Caterpillar C_1; (b) Caterpillar C_2; (c) A placement of the vertices of C_S not respecting the embedding constraints of $\mathcal{E}_2$. The polygon P is drawn with dotted segments, the edges of C_1 (of C_2) are drawn as solid (dashed) segments; (d) Placing vertices $k+1$ and $k+4$ inside P leads to an intersection between edges $(k+1, k+3)$ and $(k+2, k+4)$ of C_S; (e) Placing vertices $k+1$ and $k+4$ outside P leads to an intersection between edges $(k+2, k+4)$ and either $(k+1, k+3)$ or $(k+3, k+5)$ of C_S.

Construct a straight-line drawing Γ_1 of C_1. The embedding $\mathcal{E}_1$ of C_1 forces the vertices $1, 2, \ldots, 18$ to appear in this order around r in Γ_1. Consider the subtrees of C_1 induced by the vertices $r, 1, 2, \ldots, 6$, by the vertices $r, 7, 8, \ldots, 12$, and by the vertices $r, 13, 14, \ldots, 18$. Since such subtrees appear consecutively around r, then at least one of them must be drawn in a wedge rooted at r and with angle less than π. Let C_S be such a subtree and let $k, k+1, \ldots, k+5$ be the vertices of C_S, with $k = 1, 7$ or 13. Without loss of generality, let r be the uppermost point of this wedge. It follows that C_S must be drawn *downward*. Denote by P the polygon composed of the edges (r, k) and $(r, k+5)$ of C_1 and of the edges $(k, k+2)$, $(k+2, k+3)$, and $(k+3, k+5)$ of C_2. Note that vertices $k+1$ and $k+4$ must be either both inside or both outside P. In fact, placing one of these vertices inside and the other outside P is not consistent with the embedding constraints of $\mathcal{E}_2$; see Fig. 4(c). If both vertices $k+1$ and $k+4$ are placed inside P, then the embedding constraints of $\mathcal{E}_1$ and $\mathcal{E}_2$ and the upwardness of C_S imply that edge $(k+2, k+4)$ must cut edge $(r, k+3)$ and that edge $(k+1, k+3)$ must cut edge $(r, k+2)$. It follows that there is an intersection between edges $(k+2, k+4)$ and $(k+1, k+3)$, both belonging to C_S; see Fig. 4(d). Similarly, if both vertices $k+1$ and $k+4$ are placed outside P, then by the embedding constraints of $\mathcal{E}_1$ and $\mathcal{E}_2$ vertex $k+2$ is placed inside the polygon formed by the edges $(r, k+1)$, $(r, k+5)$ of C_1 and by the edges $(k+1, k+3)$, $(k+3, k+5)$ of C_2. Hence, edge $(k+2, k+4)$ cuts such a polygon either in edge $(k+1, k+3)$ or in edge $(k+3, k+5)$; see Fig. 4(e) and this concludes the proof. □

4.3 Simultaneous Drawing of Outerplanar Graphs and Paths with Fixed Embedding

Let O^* be the outerplanar graph on seven vertices v_1, v_2, …, v_7 shown in Fig. 5(a) and $\mathcal{E}^*$ be the embedding of O^* shown in Fig. 5(b). Let F^* be the face of $\mathcal{E}^*$ with incident vertices v_1, v_3, and v_7

and let P^* be the path $(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$. The proof of the following lemma can be found in the Appendix.



(a) (b) (c) (d)

Fig. 5. (a) Outerplanar graph O^*, drawn with solid edges, and path P^*, drawn with dashed edges. (b) Embedding $\mathcal{E}^*$ of O^*. (c) Outerplanar graph O, drawn with solid edges, and path P, drawn with dashed edges. (d) Embedding $\mathcal{E}$ of O.

Lemma 2. *There does not exist a geometric simultaneous embedding of O^* and P^* in which the embedding of O^* is $\mathcal{E}^*$ and the external face of O^* is F^*.*

Theorem 5. *There exist an outerplanar graph O, an embedding $\mathcal{E}$ of O, a path P, and a mapping between their vertices such that: (i) O and P do not share edges, and (ii) O and P have no geometric simultaneous embedding.*

Proof: Let O_1^* and O_2^* be two copies of the outerplanar graph O^* defined above. O_1^* and O_2^* have seven vertices each, and we denote by v_i^j, with $j = 1, 2$ and $i = 1, \dots, 7$, the vertex of O_j^* that corresponds to vertex v_i of O^* in O. Let $\mathcal{E}_1^*$ and $\mathcal{E}_2^*$ be the embeddings of O_1^* and O_2^* that correspond to the embedding $\mathcal{E}^*$ of O^*. Let O be the graph composed of O_1^*, of O_2^*, and of the edges (v_7^1, v_1^2), (v_1^1, v_7^2); see Fig. 5(c). Let the embedding $\mathcal{E}$ for O be defined as follows: (i) each vertex of O_1^* (of O_2^*) but for v_1^1 and v_7^1 (but for v_1^2 and v_7^2) has the same adjacency list as in $\mathcal{E}_1^*$ (in $\mathcal{E}_2^*$); (ii) the adjacency lists of the remaining vertices are as follows:$v_1^1 \rightarrow (v_7^1, v_6^1, v_4^1, v_3^1, v_7^2)$, $v_7^1 \rightarrow (v_1^2, v_3^1, v_2^1, v_5^1, v_1^1)$, $v_1^2 \rightarrow (v_7^2, v_6^2, v_4^2, v_3^2, v_7^1)$, $v_7^2 \rightarrow (v_1^1, v_3^2, v_2^2, v_5^2, v_1^2)$.

As the embedding of O is fixed, the faces of a planar drawing of O with embedding $\mathcal{E}$ are fully determined up to the choice of the outerface. Let F_1^* and F_2^* denote the cycles (v_1^1, v_3^1, v_7^1) and (v_1^2, v_3^2, v_7^2), respectively. Note that these cycles are faces of O_1^* and O_2^*. Let P be the path $(v_7^1, v_6^1, v_5^1, v_4^1, v_3^1, v_2^1, v_1^1, v_1^2, v_2^2, v_3^2, v_4^2, v_5^2, v_6^2, v_7^2)$. It is easy to verify that O and P do not share edges. Further, the subpaths of P induced by the vertices of O_1^* (O_2^*) play for O_1^* (O_2^*) the same role that path P^* plays for graph O^* in Lemma 2.

We now show that every plane drawing $\Gamma_{\mathcal{E}}$ of O with embedding $\mathcal{E}$ determines a non-planar drawing of P. Consider a particular plane embedding $\mathcal{E}_O$ of O obtained by choosing $(v_1^1, v_7^1, v_1^2, v_7^2)$ as external face; see Fig. 5(d). Now observe that choosing any face internal to F_1^* (F_2^*) in $\mathcal{E}_O$ as the external face of $\Gamma_{\mathcal{E}}$ leaves O_2^* (O_1^*) embedded with external face F_2^* (F_1^*) and that choosing any face external to both F_1^* and F_2^* in $\mathcal{E}_O$ as the external face of $\Gamma_{\mathcal{E}}$ leaves O_1^* and O_2^* embedded with external faces F_1^* and F_2^*, respectively. Hence, we can apply Lemma 2 and conclude that there is no simultaneous embedding of O and P. □

5 Near-Simultaneous Embedding

In this section we study the variation of geometric simultaneous embedding in which vertices that represent the same entity in different graphs are allowed to be placed in different points in the different drawings. The relaxation of the constraint that forces vertices to be placed exactly in the same point should allow us to near-simultaneously embed larger classes of graphs. However, in order to preserve the viewer's "mental map" corresponding vertices should be placed as close as possible. This turns out to be impossible for general planar graphs, as the first lemma of this section shows. First, define the *displacement* of a vertex v between two drawings Γ_1 and Γ_2 as the distance between the location of v in Γ_1 and the location of v in Γ_2. Second, we show that there exist two n-vertex planar graphs G_1 and G_2 with a bijection γ between their vertices such that for any two planar straight-line grid drawings Γ_1 and Γ_2 of G_1 and G_2, respectively, there exists a vertex v that has a displacement $\Omega(n)$ between Γ_1 and Γ_2.

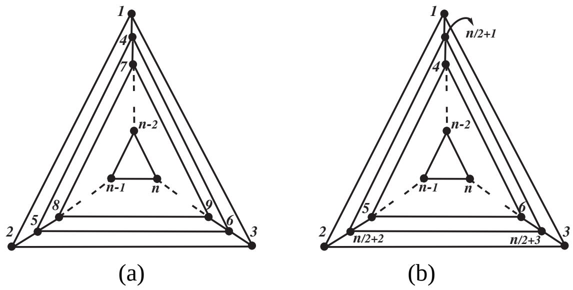


Fig. 6. (a) Nested triangle graph G_1; (b) Nested triangle graph G_2.

Let G_1 and G_2 be two *nested triangle* graphs, each on n vertices; see Fig. 6. A nested triangle graph G is a triconnected planar graph with triangular face $F(G)$ such that removing the vertices of $F(G)$ and their incident edges leaves a smaller nested triangle graph or an empty vertex set. Suppose the mapping $\gamma(v_1) = v_2$ between vertices $v_1 \in V(G_1)$ and vertices $v_2 \in V(G_2)$ is the one shown in Fig. 6. Formally, the mapping can be defined by the following procedure: embed G_1 and G_2 with external faces $F(G_1)$ and $F(G_2)$, respectively. Starting from G_1 (G_2), for $i = 1, \ldots, n/3$, remove from the current graph the three vertices of the external face and label them $3i - 2, 3i - 1$, and $3i$ ($3(i+1)/2 - 2, 3(i+1)/2 - 1$, and $3(i+1)/2$ if i is odd, or $(n+3i)/2 - 2, (n+3i)/2 - 1$, and $(n+3i)/2$ if i is even). Then, for any two planar straight-line grid drawings Γ_1 of G_1 and Γ_2 of G_2 and G_2, we have:

Lemma 3. *There exists a vertex representing the same entity in G_1 and G_2 that has displacement $\Omega(n)$ between Γ_1 and Γ_2.*

Due to space constraints we leave the proof of the previous lemma in the Appendix. The lower bound in Lemma 3 concerning the distance between two consecutive placements of a vertex in two different drawings is easily matched by an upper bound obtained by independently drawing each planar graph in $O(n) \times O(n)$ area: Each vertex is displaced by at most the length of the diagonal of the drawing's bounding box. Clearly, such a diagonal has length $O(n)$.

The above result shows that we cannot hope to guarantee near-simultaneous embeddings for arbitrary pairs of planar graphs. It is possible, however, that for graphs that are "similar", near-simultaneous embeddings might exist. Similarity between graphs could be defined and regarded in several different ways, by minding both the combinatorial structure of the graphs and the mapping between the vertices of the graphs. With this in mind, in the following we look for near-simultaneous embeddings of similar paths and similar trees.

5.1 Near-simultaneous drawings of similar paths

Recall that two arbitrary paths always have a geometric simultaneous embedding, while three of them might not have one [1]. Therefore, in order to represent a sequence of paths using a sequence of planar drawings, vertices that are in correspondence under the mapping must be displaced from one drawing to the next.

Observing that a path induces an ordering of the vertices, call two n-vertex paths P_1 and P_2 with orderings π_1 and π_2 of their vertices and with a fixed bijective mapping γ between their vertices k-*similar* if for each vertex $v_1 \in P_1$ the position of v_1 in π_1 differs by at most k positions from the position of $v_2 = \gamma(v_1)$ in π_2. Then a simple drawing of the paths as parallel horizontal polygonal lines with uniform horizontal distances between adjacent vertices gives a near-simultaneous drawing. As any vertex v_i if P_1 occurs within k positions in P_2 (compared with its position in P_1) then the extent of the displacement of the vertex from one drawing to the next is limited by exactly k units. More generally, this idea can be summarized as follows:

Theorem 6. *A sequence of n-vertex paths $P_0, P_1, ..., P_m$, where each two consecutive paths are k-similar, can be drawn so that the displacement of any vertex in a pair of paths that are consecutive in the sequence is at most k.*

5.2 Near-simultaneous drawings of similar trees

Generalizing the idea of k-similarity to trees, call two (rooted) trees T_1 and T_2 with vertex sets V_1 and V_2 and with a fixed bijective mapping γ between their vertices, k-similar if:

- The depths of any vertex $v_1 \in V_1$ and of its corresponding vertex $\gamma(v_1) \in V_2$ differ by at most k;
- The positions of any two corresponding vertices in any pre-established kind of traversal of the trees in which a parent is encountered before its children (for instance in pre-, post-, in-order, or breadth-first-search) differ by at most k.

Given two trees T_1 and T_2 that are k-similar with respect to a pre-established traversal order π, we can draw each of T_1 and T_2 as follows: (1) Assign to each vertex v_i its position $\pi(v_i)$ as an x-coordinate; (2) Assign to each vertex v_i its depth as a y-coordinate.

Such an algorithm produces layouts that are planar and *layered*. A drawing is layered if (i) each vertex is assigned to a *layer*, (ii) for each layer an order of its vertices is specified, and (iii) there are only edges joining vertices on consecutive layers or joining vertices on the same layer that are consecutive in that layer's ordering. Since subsequent trees are k-similar, the depth of any vertex and its position in a tree traversal changes only by k in two consecutive trees; hence, we have that the displacement of a vertex representing the same entity in different drawings is given by $\sqrt{k^2 + k^2} = k\sqrt{2}$. This result implies that the described algorithm can be used to visualize a sequence of k-similar trees by a sequence of planar drawings, displacing each vertex by at most $k\sqrt{2}$ units from a layout to the next, giving us the following theorem.

Theorem 7. *A sequence of n-vertex trees $T_0, T_1, ..., T_m$, where each two consecutive trees are k-similar, can be drawn such that the displacement of any vertex in a pair of trees that are consecutive in the sequence is at most $k\sqrt{2}$.*

Observe that an analogous definition of similarity between two graphs and the same layout algorithm work more generally for *level planar graphs* [12,13], which are graphs that admit a planar layered drawing, by only replacing the depth of a vertex with the depth of a vertex in a spanning tree of the graph. Notice that the class of *outerplanar graphs* is enclosed in that of the level planar graphs. Finally, the area requirement of the drawings produced by the described algorithm is worst-case quadratic in the number of vertices of a tree (or of a level planar graph).

6 Conclusions

In this paper we have considered some variations of the well-known problem of embedding graphs simultaneously. Namely, we have studied the setting in which no edges are shared by the input graphs, the setting in which the input graphs have fixed embeddings, and the setting in which vertices are allowed to be placed in near points in different drawings.

Concerning the geometric simultaneous embedding without common edges, we provided a negative result that seems to show that the geometric simultaneous embedding is not more powerful by assuming the edge sets of the input graphs to be disjoint. Further, we believe that there exist two trees not sharing common edges that do not admit a geometric simultaneous embedding. This would extend the result in [11] where two trees that do not admit a simultaneous embedding and that do share edges are shown. Consider two isomorphic rooted trees $T_1(h,k)$ and $T_2(h,k)$ that do not have any edges in common and a mapping γ between their vertices, shown in Fig. 7 and defined as follows:

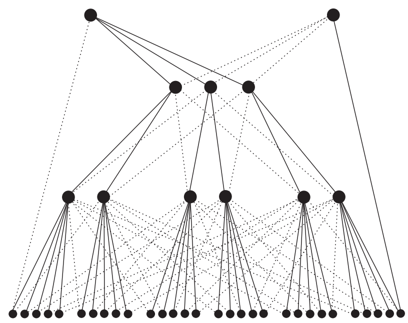


Fig. 7. Trees $T_1(3,3)$ and $T_2(3,3)$ with the mapping γ between their vertices. $T_1(3,3)$ has solid edges and $T_2(3,3)$ has dashed edges.

- the root of $T_1(h,k)$ (of $T_2(h,k)$) has k children;
- each vertex of $T_1(h,k)$ (of $T_2(h,k)$) at distance i from the root, with $1 \leq i < h$, has a number of children one less than the number of vertices at distance i from the root in $T_1(h,k)$ (in $T_2(h,k)$);
- one vertex of $T_1(h,k)$ (of $T_2(h,k)$) at distance h from the root has one child;
- each child of the root of $T_1(h,k)$ is mapped to a distinct child of the root of $T_2(h,k)$;
- for each pair of vertices v_1 of $T_1(h,k)$ and v_2 of $T_2(h,k)$ that are at distance i from the root of their own tree and that are such that $v_2 \neq \gamma(v_1)$, there exists a child of v_1 that is mapped to a child of v_2;
- the only vertex of $T_1(h,k)$ (of $T_2(h,k)$) that is at distance $h+1$ from the root is mapped to the root of $T_2(h,k)$ (to the root of $T_1(h,k)$).

Conjecture 1. For sufficiently large h and k, $T_1(h, k)$ and $T_2(h, k)$ do not admit a geometric simultaneous embedding with mapping γ between their vertices.

For the problem of drawing graphs simultaneously with fixed embedding, we provided more negative results than in the usual setting for geometric simultaneous embedding, while providing only two positive results partially covering the ones already known for geometric simultaneous embedding. We believe that understanding the possibility of obtaining a simultaneous embedding of a tree and a path in which the tree has a fixed embedding could be useful for the same problem in the non-fixed embedding setting.

Even in the more relaxed near-simultaneous setting, we have shown that without assuming a similarity in the sequence of graphs to be drawn, it is difficult to limit the displacement of a vertex from a drawing to the next. We have shown that for paths, for trees, and for level planar graphs there exist reasonable similarity measures that allow us to obtain near-simultaneous drawings. However, in the case of general planar graphs it is not yet clear what kind of similarity metric can be defined and how well can such graphs be drawn.

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Appendix: Proofs of Lemmas 2 and 3

Proof of Lemma 2

Lemma 2. There does not exist a geometric simultaneous embedding of O^* and P^* in which the embedding of O^* is $\mathcal{E}^*$ and the external face of O^* is F^*.

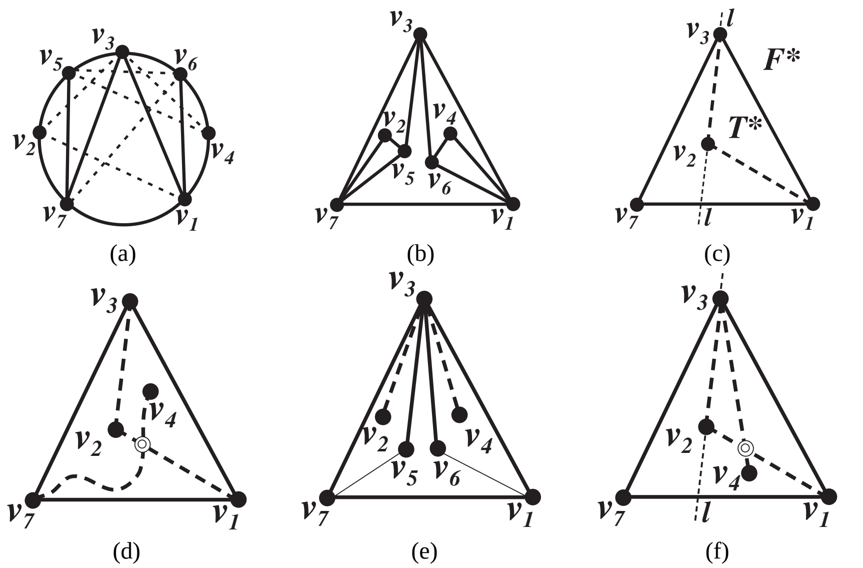

Fig. 8. (a) Outerplanar graph O^*, drawn with solid edges, and path P^*, drawn with dashed edges. (b) Embedding $\mathcal{E}^*$ of O^*; (c) Triangle T^*, face F^*, and line l. (d) Embedding vertex v_4 inside T^* creates a crossing between the subpath of P^* connecting v_1 and v_3 and the subpath of P^* connecting v_4 and v_7; (e) Order of the edges incident on vertex v_3; (f) Embedding vertex v_4 outside T^* and in the *interior part* of l creates a crossing between edges (v_1, v_2) and (v_3, v_4) of P^*.

Proof: We begin by observing that since F^* is the external face of $\mathcal{E}^*$ then all vertices of O^*, other than v_1, v_3, and v_7, are contained inside F^*. Consider the triangle T^* formed by the edges (v_1, v_2), (v_2, v_3) of P^*, and by the edge (v_1, v_3) of O^*; see Fig. 8(c). As vertex v_7 belongs to the external face of $\mathcal{E}^*$ it must be outside T^*. Let l be the line passing through v_2 and v_3; l separates the plane in two open half-planes, one containing v_7, called the *exterior part* of l, and one not containing v_7, called the *interior part* of l. We show that every placement of v_4 leads to a crossing in the drawing of the path, if the drawing of O^* respects the given embedding and has no crossings.

If v_4 is placed inside T^*, then the subpath of P^* composed of the edges (v_1, v_2) and (v_2, v_3) crosses the subpath of P^* connecting v_4, that lies inside T^*, and v_7, that lies outside T^*; see Fig. 8(d). Suppose v_4 is placed outside T^*. By the embedding constraints of $\mathcal{E}$ and by the observation that vertex v_4 (vertex v_2) must lie inside triangle $\Delta v_1 v_3 v_6$ (inside triangle $\Delta v_3 v_5 v_7$), the clockwise order of the edges of O^* and of P^* incident in v_3 is $(v_3, v_1), (v_3, v_4), (v_3, v_6), (v_3, v_5), (v_3, v_2), (v_3, v_7)$; see Fig. 8(e). Therefore, v_4 must be in the *interior part* of l and this implies that edge (v_1, v_2) crosses edge (v_3, v_4) in P^*; see Fig. 8(f). □

Proof of Lemma 3

Lemma 3. There exists a vertex representing the same entity in G_1 and G_2 that has displacement $\Omega(n)$ between Γ_1 and Γ_2.

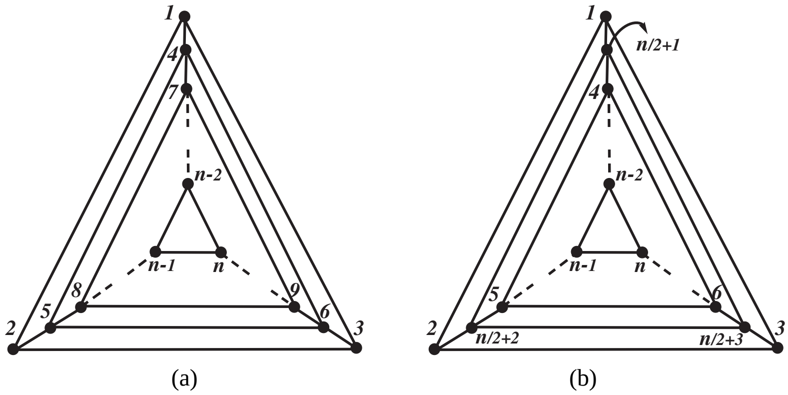

Fig. 9. (a) Nested triangle graph G_1. (b) Nested triangle graph G_2.

Proof: A nested triangle graph is triconnected, so the only degree of freedom for obtaining a plane embedding of such a graph is given by the choice of its external face. By choosing any external face f for a nested triangle graph G that is formed by t nested triangles, two nested triangles structures $T_1(G)$ and $T_2(G)$ with triangular external face are present, one with t_1 and the other with t_2 triangles, with $0 \leq t_1, t_2 \leq t$ and with $t_1 + t_2 = t$. In a plane embedding of G with external face f, $T_1(G)$ and $T_2(G)$ cover disjoint portions of the plane. Hence, Γ_1 (Γ_2) has two nested triangles structures $T_1(G_1)$ and $T_2(G_1)$ (resp. $T_1(G_2)$ and $T_2(G_2)$). Note that $T_1(G_1)$ or $T_2(G_1)$ (resp. $T_1(G_2)$ or $T_2(G_2)$) might be without vertices. It is easy to see that there exist indices i and j, with $i, j \in \{1, 2\}$, such that $T_i(G_1)$ and $T_j(G_2)$ share a linear number of vertices.

In a grid drawing of a nested triangle graph in which the external face is chosen to be a triangular face, if two vertices v_1 and v_2 belong to two different triangles that are separated by t triangles in the nested structure, then the x-coordinate or the y-coordinate of the two vertices differs by at least t units. Consider the sub-drawing of Γ_2 corresponding to the subgraph T^* of $T_j(G_2)$ made of $k = an+b$ most deeply nested triangles of $T_j(G_2)$, with a and b constants. Note that such a subgraph has a fixed plane embedding with outer face O. Choose a and b so that the vertices incident on O belong also to $T_i(G_1)$. Now consider the three most deeply nested triangles T_1, T_2, and T_3 of T^*, such that T_1 is nested inside T_2 that is nested inside T_3. We now have two cases to consider:

- If there is a vertex v of T_1, of T_2, or of T_3 that does not belong to $T_i(G_1)$, then it will be embedded outside T^* in Γ_1. Since T^* is made of $O(n)$ nested triangles, v is mapped in Γ_2 into a point at distance $\Omega(n)$ from the point where v is mapped in Γ_1.
- Otherwise every vertex of T_1, T_2, and T_3 belongs to $T_i(G_1)$. Note that the labels of the vertices of T_1 (of T_2) differ from the labels of the vertices of T_2 (resp. of T_3) by at least $\frac{n}{2} - 5$ units and that in $T_i(G_1)$ there are $O(n)$ triangles separating T_1 and T_2. This implies that either the position of the vertices of T_1, or the position of the vertices of T_2, or the position of the vertices of T_3 in Γ_1 and in Γ_2 is at distance $\Omega(n)$.

□