Characterizing Simultaneous Embedding with Fixed Edges

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Abstract

A set of planar graphs share a simultaneous embedding if they can be drawn on the same vertex set V of n vertices in the plane without crossings between edges of the same graph. Fixed edges are common edges between graphs that share the same Jordan curve in the simultaneous drawing. We give a necessary condition for when pairs of graphs can have a *simultaneous embedding with fixed edges* (SEFE). This allows us to determine for which (outer)planar graphs always have a SEFE with all (outer)planar graphs with $O(n^2 \lg n)$ time drawing algorithms. This allows us to decide in $O(n)$ time whether a pair of biconnected outerplanar graphs has a SEFE.

1 Introduction

While the union of any pair of edge sets may be nonplanar, a planar drawing of each layer may be possible. Crossings between edges of distinct edge sets are allowed. This is the problem of *simultaneous embedding* (SE) that generalizes the notion of planarity among multiple graphs [1]. While any number of planar graphs can be drawn on the same fixed set of vertex locations, difficulties arise once straight-line edges are required. This is the problem of *simultaneous geometric embedding* (SGE). If edge bends are allowed, then having common edges drawn in the same way preserves the "mental map". Such edges are *fixed edges* leading to the problem of *simultaneous embedding with fixed edges* (SEFE). Since straight-line edges between a pair of vertices are also fixed, any graph that has a SGE also has a SEFE, but the converse is not true, i.e, SEFE $\subset$ SGE $\subset$ SE. Deciding if two graphs have a SGE is NP-hard [2], whereas,

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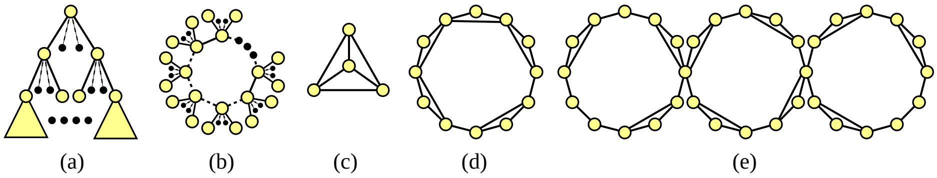

Fig. 1. Graphs (a)–(c) have a SEFE with any planar graph, whereas, (d)–(e) have a SEFE with any outerplanar graph.

deciding if three graphs have a SEFE is NP-complete [4]. However, deciding if two graphs have a SEFE in polynomial time remains open.

1.1 Our Contribution

We omit proofs of claims, but details can be found in the technical report [3].

(i) While most pairs of graphs whose union forms a subdivided K_5 or $K_{3,3}$ share a SEFE, we provide 16 minimal forbidden pairs that do not. This leads to a necessary condition for when two graphs can have a SEFE.

(ii) We show that the only graphs that *always* have a SEFE with *any* planar graph are (i) forests, (ii) circular caterpillars, and (iii) K_4 subgraphs. Likewise, we show that the only outerplanar graphs that *always* have a SEFE with *any* outerplanar graph are (i) biconnected such that the endpoints of each chord are at a distance 2 along the outerface (a K_3-*cycle*) or (ii) have a cubic K_3-cycle for each biconnected subgraph; see Fig. 1.

(iii) Using a forbidden outerplanar pair, we give a linear time decision algorithm for deciding if two biconnected outerplanar graphs have a SEFE.

2 Forbidden SEFE Pairs

Suppose $G_1(V, E_1)$ and $G_2(V, E_2)$ have no SEFE as in Fig. 2(a). If deleting any edge allows a SEFE, then G_1 and G_2 are *edge minimal* as in Fig. 2(b). If vertex v (adjacent to u and w) is degree-2 in the union $G_1 \cup G_2$, but not degree-1 in G_1 or G_2, then we can *unsubdivide* v by replacing edges (u, v) and (v, w) with (u, w). A pair for which this cannot be done is *vertex minimal* as in Fig. 2(c). A *minimal forbidden pair* is edge and vertex minimal and does not have a SEFE. An *alternating edge* is a $u \rightsquigarrow v$ path in which the edges strictly alternate between being in either G_1 or G_2, but not both. An *exclusive edge* is an edge (u, v) that is only in G_1 or G_2, while an *inclusive edge* is a fixed edge (u, v) in the intersection $G_1 \cap G_2$.

Claim 2.1 *Any pair $G_1(V, E_1)$ and $G_2(V, E_2)$ is reducible to a pair where every $u \rightsquigarrow v$ path is either an incident, an exclusive or an alternating edge.*

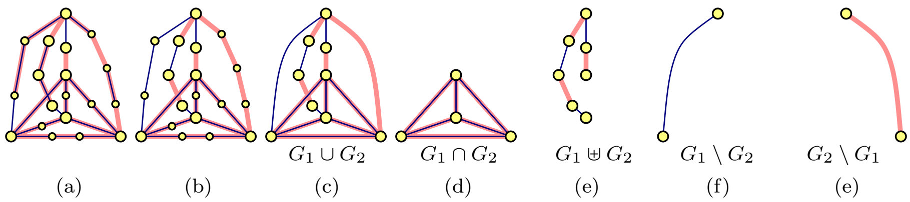

Fig. 2. Removing edges and vertices from (a) gives (b) and (c) with subgraphs (d)–(g).

A pair G_1 and G_2 in which all $u \rightsquigarrow v$ paths have been reduced is called a *reduced pair*. We define $G_1 \uplus G_2$ as the $G_1 \cup G_2$ subgraph of alternating edges, and $G_1 \setminus G_2$ as the G_1 subgraph of exclusive edges.

Lemma 2.2 *Suppose the union of reduced pair G_1 and G_2 is homeomorphic to K_5 or $K_{3,3}$. Let $u \rightsquigarrow v$ and $x \rightsquigarrow y$ be nonincident paths in $G_1 \cup G_2$ but not in $G_1 \cap G_2$. G_1 and G_2 share a SEFE if either path belongs to $G_1 \uplus G_2$ or one belongs to $G_1 \setminus G_2$ and the other belongs to $G_2 \setminus G_1$.*

This allows us to determine when a pair forming K_5 or $K_{3,3}$ has a SEFE.

Corollary 2.3 *Suppose the union of reduced pair G_1 and G_2 is homeomorphic to K_5 or $K_{3,3}$. G_1 and G_2 have no SEFE if and only if (i) every nonincident edge of an alternating edge in $G_1 \uplus G_2$ is in $G_1 \cap G_2$ and (ii) every nonincident edge of an exclusive edge in $G_1 \setminus G_2$ is in G_1.*

From this we can produce a necessary, but insufficient, condition for SEFE.

Theorem 2.4 *There are 16 minimal forbidden pairs with a union homeomorphic to K_5 or $K_{3,3}$.*

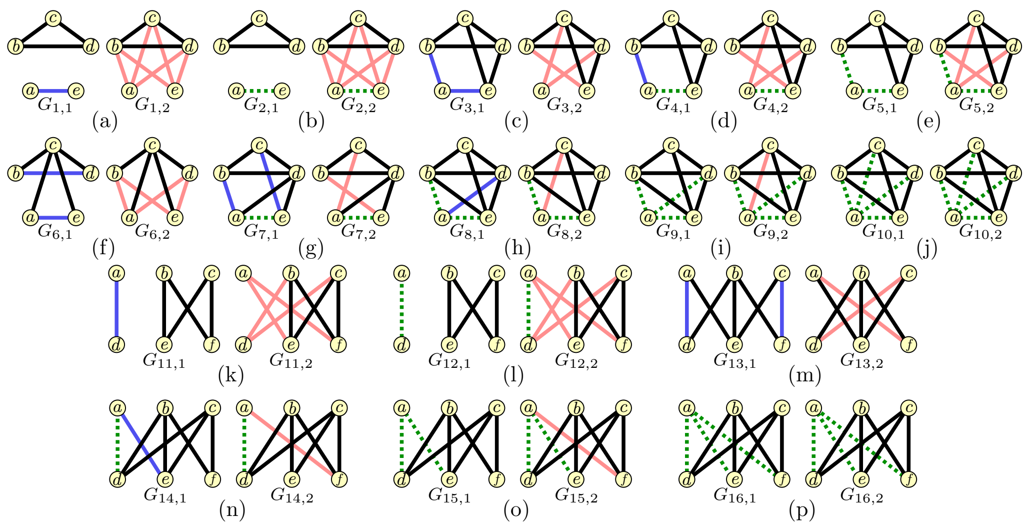

Fig. 3. Sixteen minimal forbidden pairs. The dashed edges are in $G_1 \uplus G_2$.

3 Characterizing and Deciding SEFE

Next we determine which (outer)planar graphs *always* have a SEFE with *any* (outer)planar graph. Let $\mathcal{P}$ be the set of planar graphs and $\mathcal{P}_{\mathsf{SEFE}} \subset \mathcal{P}$ be the subset of $\mathcal{P}$ consisting of forests, circular caterpillars, and subgraphs of K_4.

Theorem 3.1 *$\mathcal{P}_{\mathsf{SEFE}}$ are the only planar graphs that always have a SEFE with any planar graph. Each planar graph in $\mathcal{P}_{\mathsf{SEFE}}$ can be drawn in $O(n^2 \lg n)$ time.*

A K_3*-cycle* is a biconnected outerplanar graph such that the endpoints of each chord are at a distance 2 along the outerface. Let $\mathcal{O}$ be the set of outerplanar graphs and $\mathcal{O}_{\mathsf{SEFE}} \subset \mathcal{O}$ be the subset of $\mathcal{O}$ consisting of K_3-cycles and outerplanar graphs with cubic K_3-cycles for each biconnected subgraph.

Theorem 3.2 *$\mathcal{O}_{\mathsf{SEFE}}$ are the only outerplanar graphs that always have a SEFE with any outerplanar graph. Each outerplanar graph in $\mathcal{O}_{\mathsf{SEFE}}$ can be drawn in $O(n^2 \lg n)$ time.*

While Theorem 2.4 is not sufficient in general, we can show sufficiency for the more restrictive case of pairs of biconnected outerplanar graphs.

Theorem 3.3 *Let (G_1, G_2) be a pair of biconnected outerplanar graphs. Pair (G_1, G_2) has a SEFE if and only if the pair does not have the minimal forbidden pair $(G_{13,1}, G_{13,2})$. This can be decided in $O(n)$ time.*

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