Upward Straight-Line Embeddings of Directed Graphs into Point Sets

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Abstract. In this paper we consider the problem of characterizing the directed graphs that admit an upward straight-line embedding into every point set in convex or in general position. In particular, we show that no biconnected directed graph admits an upward straight-line embedding into every point set in convex position, and we provide a characterization of the Hamiltonian directed graphs that admit upward straight-line embeddings into every point set in general or in convex position. We also describe how to construct upward straight-line embeddings of directed paths into convex point sets and we prove that for directed trees such embeddings do not always exist. Further, we investigate the related problem of upward simultaneous embedding without mapping, proving that deciding whether two directed graphs admit an upward simultaneous embedding without mapping is $\mathcal{NP}$-hard.

1 Introduction

A *straight-line embedding* of a graph into a point set P is a mapping of each vertex to a point of P and of each edge to a straight-line segment between its end-points such that no two edges intersect. The problem of constructing straight-line embeddings of graphs into planar point sets is well-studied from both a combinatorial and an algorithmic point of view and comes in several different flavours within the Graph Drawing literature.

Gritzmann *et al.* [12] proved that a graph admits a straight-line embedding into *every* point set in general position if and only if it is an *outerplanar graph*. From an algorithmic point of view, an $O(n \log^3 n)$-time algorithm [1] and a $\Theta(n \log n)$-time algorithm [2] are known for constructing straight-line embeddings of outerplanar graphs and trees into given point sets in general position, respectively. Cabello [4] proved that the problem of deciding whether a planar graph admits a straight-line embedding into a given point set is $\mathcal{NP}$-hard. If edges are not required to be straight, then by the results of Kaufmann and Wiese [13], every planar graph admits a planar drawing with at most two bends per edge into every point set and such a bound can not be improved.

Several problems ask for embedding more graphs on the same point set. Determining the minimum cardinality $f(n)$ of a set of points P such that every n-vertex planar graph admits a straight-line embedding into an n-point subset of P is a well-know and widely-open problem [5,6,14,15]. Recently, the problem of constructing simultaneous embeddings without mapping, i.e., straight-line planar drawings of n-vertex graphs on

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the same set of n points, has been considered. Brass *et al.* [3] proved that a planar graph and any number of outerplanar graphs admit a simultaneous embedding without mapping. Whether every two planar graphs admit a simultaneous embedding without mapping is still unknown.

Surprisingly, less attention has been devoted to the directed versions of such problems. When visualizing a directed graph, one usually requires an *upward* drawing, i.e., a drawing such that each edge monotonically increases in the y-direction. Giordano *et al.* [11] show a directed counterpart of the results in [13], namely that every upward planar digraph has an upward planar embedding with at most two bends per edge into every point set. As a directed counterpart of the results in [3], the same authors show that any number of trees admit an upward simultaneous embedding without mapping.

In this paper we study additional directed versions of some of the problems cited above. In Section 3, we consider the problem of determining which directed graphs admit a planar straight-line upward embedding into every point set in general or in convex position. We show that no biconnected directed graph with more than three vertices has a straight-line upward embedding into every point set in convex position. We also characterize the Hamiltonian directed graphs that admit a straight-line upward embedding into every point set in convex or in general position. We prove that every directed path admits a straight-line upward embedding into every point set in convex position and that every directed tree of diameter at most four admits a straight-line upward embedding into every point set in general position. Finally, we prove that not all directed trees admit a straight-line upward embedding into every point set in convex position.

In Section 4, answering a question of Giordano *et al.* [11], we show two upward planar directed graphs that do not admit any upward simultaneous embedding without mapping. Further, we prove that deciding if two upward planar digraphs admit an upward simultaneous embedding without mapping is $\mathcal{NP}$-hard.

Several proofs are omitted or sketched, due to space limitations. Full proofs of each statement can be found in the extended version of the paper [9].

2 Preliminaries

An *upward planar directed graph* is a directed graph that admits a planar drawing such that each edge is represented by a curve monotonically increasing in the y-direction. Every upward planar digraph admits a *straight-line* upward planar drawing [7], i.e., an upward planar drawing in which every edge is represented by a segment. The *underlying graph* of a directed graph G is the undirected graph obtained by removing the directions on the edges of G. In the following we refer to directed paths, directed cycles, directed trees, directed outerplanar graphs, meaning upward planar directed graphs whose underlying graphs are paths, cycles, trees, and outerplanar graphs, respectively. A *Hamiltonian* directed graph G is a directed graph containing a path $(v_1, v_2, \cdots, v_n)$ passing through all vertices of G such that edge (v_i, v_{i+1}) is directed from v_i to v_{i+1}, for each $1 \leq i < n$. A *source* (resp. *sink*) in a directed graph G is a vertex having only outgoing edges (resp. having only incoming edges).

A set of points in the plane is in *general position* if no three points lie on the same line. The *convex hull* of a set of points P is the set of points that can be obtained as a convex combination of the points of P. A set of points is in *convex position* if no point is in the convex hull of the others.

An *upward straight-line embedding* of an n-vertex directed graph G into a set P of n points is a mapping of each vertex of G to a point of P providing a straight-line upward planar drawing. An *upward simultaneous embedding without mapping* of two n-vertex upward planar directed graphs G_1 and G_2 is a pair of upward planar straight-line drawings in which the vertices of G_1 and G_2 are placed on the same set of n points.

In order to deal with upward embeddings of directed graphs into point sets, we assume that no two points of any point set have the same y-coordinate[1]. Then, the points of any point set can be totally ordered by increasing y-coordinate. Hence, we refer to the i-th point as to the point such that exactly $i-1$ points have smaller y-coordinate. The first and the last point of a point set P are denoted by $p_m(P)$ and $p_M(P)$, respectively. In a convex point set P, two points are *adjacent* if the segment between them is on the border of the convex hull of P. Points $\{v_1, v_2, \cdots, v_k\}$ in a convex point set P are *consecutive* if v_i and v_{i+1} are adjacent, for each $1 \leq i < k$. We call a *one-side* convex point set any convex point set P in which $p_M(P)$ and $p_m(P)$ are adjacent.

3 Graphs with Upward Straight-Line Embeddings into Every Point Set

In this section we consider the problem of determining which directed graphs admit an upward straight-line embedding into every point set in general or in convex position.

3.1 Biconnected Directed Graphs and Hamiltonian Directed Graphs

First, we show that no biconnected directed graph with more than three vertices has an upward straight-line embedding into every point set in convex position, and hence into every point set in general position. The following two lemmata are well-known:

Lemma 1. *Let C be any directed cycle. The number of sources in C is equal to the number of sinks.*

Lemma 2. *Let O be a straight-line embedding of a directed graph into a point set in convex position. Then O is an outerplanar embedding.*

We now observe the following lemmata.

Lemma 3. *Let G be a directed graph containing a cycle C. Suppose that C has at least two vertices u and v that are sources in C. Then there exists a convex point set P such that G has no upward straight-line embedding into P.*

[1] Such an assumption is not a great loss of generality as every point set can be turned into one without two points having the same y-coordinate, by rotating the Cartesian axes by an arbitrarily-small angle. Further, assuming that no two points have the same y-coordinate avoids trivial counter-examples and the *a priori* impossibility of drawing an edge between two specified points of the point set.

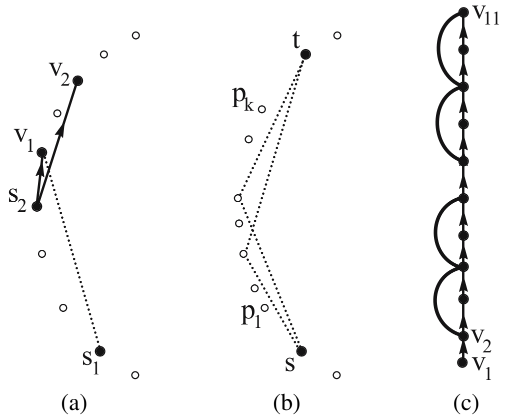


Fig. 1. (a)–(b) Illustrations for the proofs of Lemmata 3 and 4, respectively. Dotted segments represent paths connecting two vertices. (c) A graph belonging to the family $\mathcal{G}_{11}$ of Hamiltonian directed graphs with 11 vertices that admit upward straight-line embeddings into every point set.

Proof. Consider any one-side convex point set P. Let s_1 and s_2 be two sources in C. Suppose, w.l.o.g., that the point on which s_1 is drawn has y-coordinate smaller than the one on which s_2 is drawn. Since s_2 is a source in C, there exist edges (s_2, v_1) and (s_2, v_2) going out of s_2 and belonging to C. Suppose, w.l.o.g., that the point on which v_1 is drawn has y-coordinate smaller than the one on which v_2 is drawn. Since C is a cycle, there exist two disjoint paths connecting v_1 and s_1. One of these paths does not contain s_2 and v_2 and hence it crosses edge (s_2, v_2); see Fig. 1.a. □

Lemma 4. *Let G be a directed graph containing a cycle C. Suppose that C has exactly one source s and one sink t. Suppose also that each of the two directed paths $\mathcal{P}_1$ and $\mathcal{P}_2$ of C connecting s and t has at least one node different from s and t. Then there exists a convex point set P such that G has no upward straight-line embedding into P.*

Proof. Consider any one-side convex point set P. Consider any drawing Γ of G into P and let $p(s)$ and $p(t)$ be the points of P where s and t are drawn in Γ, respectively. Consider the subset P_k of P whose points have y-coordinates greater or equal than the one of $p(s)$ and less or equal than the one of $p(t)$. Since Γ is straight-line and upward, both $\mathcal{P}_1$ and $\mathcal{P}_2$ lie inside the convex hull H_k of P_k. As $\mathcal{P}_1$ and $\mathcal{P}_2$ touch the border of H_k in at least one point different from $p(s)$ and $p(t)$, $\mathcal{P}_1$ and $\mathcal{P}_2$ cross; see Fig. 1.b. □

Theorem 1. *There exists no biconnected directed graph with more than three vertices that admits an upward straight-line embedding into every point set in convex position.*

Proof. Consider any biconnected directed graph G with more than three vertices. Consider any one-side convex point set P_1. By Lemma 2, any embedding Γ of G into P_1 is outerplanar, hence all vertices of G are incident to the outer face of G. Since G is biconnected, the outer face of G is a simple cycle C. Hence, C is a cycle passing through all vertices of G. By Lemma 3, C contains exactly one vertex s that is source in C and hence, by Lemma 1, exactly one vertex t that is sink in C. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the directed paths of C connecting s and t. By Lemma 4, one out of $\mathcal{P}_1$ and $\mathcal{P}_2$, say $\mathcal{P}_1$, is a Hamiltonian directed path and the other one, say $\mathcal{P}_2$, is edge (s,t). Now consider any convex point set P_2 in which the line l through $p_m(P_2)$ and $p_M(P_2)$ determines two half-planes both containing points of P_2. Such a point set exists if $n \geq 4$. Since $\mathcal{P}_1$ is a Hamiltonian directed path between s and t, there is a vertex of $\mathcal{P}_1$ in each point of P_2. Then, there is at least one edge of $\mathcal{P}_1$ crossing l and hence crossing (s,t). □

Next, we characterize those Hamiltonian directed graphs that admit an upward straight-line embedding into every point set in general position and into every point set in convex position. Let $\mathcal{P}_n = (v_1, v_2, \cdots, v_n)$ be an n-vertex directed path, where edge (v_i, v_{i+1}) is directed from v_i to v_{i+1}, for $1 \leq i \leq n-1$. Let $\mathcal{G}_n$ be the family of n-vertex Hamiltonian directed graphs defined as follows: Each graph $G \in \mathcal{G}_n$ can be obtained by adding to $\mathcal{P}_n$ a set of edges E, where each edge of E is directed from a vertex v_i to a vertex v_{i+2}, for some $1 \leq i \leq n-2$, and no two edges (v_i, v_{i+2}) and (v_{i+1}, v_{i+3}) belong to E, for any $1 \leq i \leq n-3$; see Fig. 1.c.

Theorem 2. *An n-vertex Hamiltonian directed graph admits an upward straight-line embedding into every point set in general position if and only if it belongs to $\mathcal{G}_n$.*

Proof. First, we prove the necessity. Suppose that there exists an Hamiltonian directed graph G that admits an upward straight-line embedding into every point set and that does not belong to $\mathcal{G}_n$. If $\mathcal{P}_n = (v_1, v_2, \cdots, v_n)$ is the Hamiltonian directed path of G, then either G contains an edge (v_i, v_j), with $i+2 < j \leq n$, for some $1 \leq i \leq n-3$, or it contains two edges (v_i, v_{i+2}) and (v_{i+1}, v_{i+3}), for some $1 \leq i \leq n-3$.

Suppose G contains edge (v_i, v_j), with $i+2 < j \leq n$. Consider any convex point set P with the following property. Let l be the line through the i-th and the j-th point of P. Then the $(i+1)$-th point and the $(i+2)$-th point of P are on different sides of l. For the upward constraint, the k-th vertex of $\mathcal{P}_n$ is drawn on the k-th point of P, for $k = 1, \cdots, n$. Hence, edge (v_{i+1}, v_{i+2}) crosses edge (v_i, v_j). Suppose G contains two edges (v_i, v_{i+2}) and (v_{i+1}, v_{i+3}), for some $1 \leq i \leq n-3$. Consider any one-side convex point set P. For the upward constraint, the k-th vertex of $\mathcal{P}_n$ is drawn on the k-th point of P, for $k = 1, \cdots, n$. Then, edge (v_i, v_{i+2}) crosses edge (v_{i+1}, v_{i+3}).

We prove the sufficiency. Consider any n-vertex Hamiltonian directed graph G in $\mathcal{G}_n$ and any point set P in general position. Draw $\mathcal{P}_n = (v_1, v_2, \cdots, v_n)$ into P as a y-monotone path. Draw edges (v_i, v_{i+2}) belonging to G. Since the drawing is straight-line, edge (v_i, v_{i+2}) intersects or overlaps only those edges intersecting the open horizontal strip S delimited by the horizontal lines through v_i and v_{i+2}. By the definition of $\mathcal{G}_n$, the only edges that have intersections with S are (v_i, v_{i+1}), (v_i, v_{i+2}) and (v_{i+1}, v_{i+2}). Since P is in general position, then v_i, v_{i+1}, and v_{i+2} are not on the same line and edges (v_i, v_{i+1}), (v_i, v_{i+2}) and (v_{i+1}, v_{i+2}) do not intersect or overlap. □

3.2 Directed Paths and Directed Trees

We show how to construct upward straight-line embeddings of directed paths into point sets in convex position. We observe the following:

Lemma 5. *Let P be any one-side convex point set with n points and let $\mathcal{P} = (v_1, v_2, \cdots, v_n)$ be any n-vertex directed path. If edge (v_1, v_2) is directed from v_1 to v_2 (resp.*

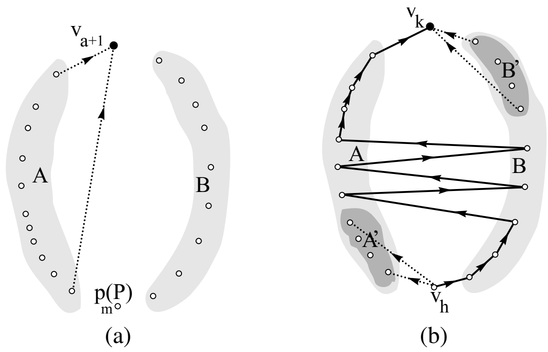


Fig. 2. (a) Illustration for the proof of Theorem 3, when (v_a, v_{a+1}) is directed from v_a to v_{a+1}, and (v_{a+1}, v_{a+2}) is directed from v_{a+2} to v_{a+1}. (b) Illustration for the proof of Theorem 3, when (v_a, v_{a+1}) is directed from v_a to v_{a+1}, and (v_{a+1}, v_{a+2}) is directed from v_{a+1} to v_{a+2}.

from v_2 to v_1), then there exists an upward straight-line embedding of $\mathcal{P}$ into P in which v_1 is on $p_m(P)$ (resp. v_1 is on $p_M(P)$).

Proof sketch. The argument uses induction on the number of points of P (and of vertices of $\mathcal{P}$). If $n = 1$ the statement trivially follows. Consider any one-side convex point set P with n points, and any n-vertex directed path $\mathcal{P} = (v_1, v_2, \cdots, v_n)$. Suppose that (v_1, v_2) is directed from v_1 to v_2, the other case being analogous. Consider the point set $P' = P \setminus \{p_m(P)\}$ which is a one-side convex point set. By induction $\mathcal{P}' = (v_2, \cdots, v_n)$ admits an upward straight-line embedding into P' in which v_2 is either on $p_M(P')$ or on $p_m(P')$ (depending on the direction of edge (v_2, v_3)). In both cases v_1 can be mapped to $p_m(P)$ and edge (v_1, v_2) can be drawn as a segment. The resulting drawing Γ is easily shown to be straight-line, upward, and planar. □

Theorem 3. *Every n-vertex directed path admits an upward straight-line embedding into every convex point set with n points.*

Proof sketch. Let $\mathcal{P} = (v_1, v_2, \cdots, v_n)$ be any directed path and let P be any convex point set with n points. Let A and B be the subsets of P to the left and to the right, respectively, of the line through $p_M(P)$ and $p_m(P)$. Let $|A| = a$ and $|B| = b$. Consider edges (v_a, v_{a+1}) and (v_{a+1}, v_{a+2}).

If edge (v_a, v_{a+1}) is directed from v_a to v_{a+1} and (v_{a+1}, v_{a+2}) is directed from v_{a+2} to v_{a+1} (see Fig. 2.a), apply Lemma 5 to construct an upward straight-line embedding of path $\mathcal{P}_1 = (v_a, v_{a-1}, \cdots, v_1)$ into A in which v_a is placed either on $p_M(A)$ or on $p_m(A)$, and apply Lemma 5 to construct an upward straight-line embedding of path $\mathcal{P}_2 = (v_{a+1}, v_{a+2}, \cdots, v_n)$ into $B \cup \{p_M(P), p_m(P)\}$ in which v_{a+1} is placed on $p_M(P)$. The resulting drawing Γ is easily shown to be straight-line, upward, and planar. An upward straight-line embedding of $\mathcal{P}$ into P can be constructed analogously if (v_a, v_{a+1}) is directed from v_{a+1} to v_a and (v_{a+1}, v_{a+2}) is directed from v_{a+1} to v_{a+2}.

If (v_a, v_{a+1}) is directed from v_a to v_{a+1} and (v_{a+1}, v_{a+2}) is directed from v_{a+1} to v_{a+2} (see Fig. 2.b), let h be the smallest index such that edge (v_i, v_{i+1}) is directed

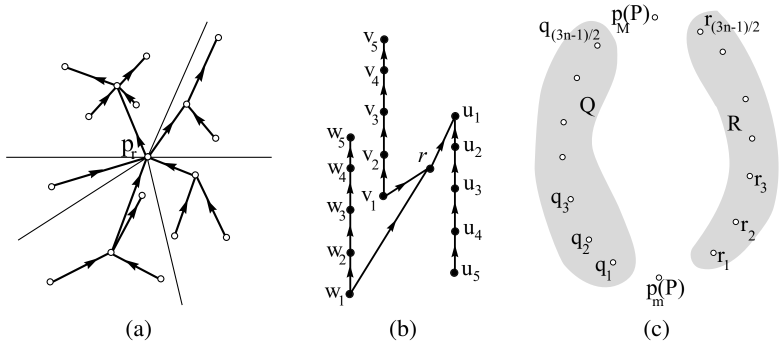


Fig. 3. (a) Illustration for Theorem 4. (b)–(c) A tree T and a point set P such that T does not admit any upward straight-line embedding into P.

from v_i to v_{i+1}, for $i = h, h+1, \ldots, a$, and let k be the greatest index such that edge (v_i, v_{i+1}) is directed from v_i to v_{i+1}, for $i = a, a+1, \ldots, k-1$. Consider path $\mathcal{P}_1 = (v_{h-1}, v_{h-2}, \cdots, v_1)$ and consider the point set $A' \subseteq A$ composed of the first $h-1$ points of A. Apply Lemma 5 to construct an upward straight-line embedding of $\mathcal{P}_1$ into A' such that v_{h-1} is placed either on $p_M(A')$, or on $p_m(A')$. Consider path $\mathcal{P}_2 = (v_{k+1}, v_{k+2}, \cdots, v_n)$ and consider the point set $B' \subseteq B$ composed of the last $n-k$ points of B. Apply Lemma 5 to construct an upward straight-line embedding of $\mathcal{P}_2$ into B' such that v_{k+1} is placed either on $p_M(B')$, or on $p_m(B')$. Consider path $\mathcal{P}_3 = (v_h, v_{h+1}, \cdots, v_k)$ and consider the point set $C' \equiv P \setminus \{A' \cup B'\}$. Construct an upward straight-line embedding of $\mathcal{P}_3$ into C' such that the i-th vertex of $\mathcal{P}_3$ is placed on the i-th point of C', for $i = h, \cdots, k$. The resulting drawing Γ is easily shown to be straight-line, upward, and planar. An upward straight-line embedding of $\mathcal{P}$ into P can be constructed analogously if (v_a, v_{a+1}) is directed from v_{a+1} to v_a, and (v_{a+1}, v_{a+2}) is directed from v_{a+2} to v_{a+1}. □

Directed trees of *diameter* at most four, i.e., in which the maximum number of edges in any path is at most four, admit upward straight-line embeddings into every point set in convex or in general position, as in the following theorem, whose constructive proof is shown in Fig. 3.a and detailed in the full version of the paper [9].

Theorem 4. *Every directed tree with diameter at most four admits an upward straight-line embedding into every point set in general position.*

Next, we show that not all directed trees admit a straight-line upward embedding into every point set in convex position (and hence into every point set in general position). The proof uses as main tool the following lemma. Consider any tree T and any convex point set P. Let u be any node of T and let $T_1, T_2, \cdots, T_k$ be the subtrees of T obtained by removing u and its incident edges from T.

Lemma 6. *In any upward straight-line embedding of T into P, the vertices of T_i are mapped into a set of consecutive points of P, for each $i = 1, 2, \cdots, k$.*

Assume n is odd and greater or equal than 5. Consider a directed tree T composed of: (i) one vertex r of degree three, (ii) three paths of n vertices $\mathcal{P}_1 = (u_1, u_2, \cdots, u_n)$, where (u_i, u_{i+1}) is directed from u_{i+1} to u_i, $\mathcal{P}_2 = (v_1, v_2, \cdots, v_n)$, where (v_i, v_{i+1}) is directed from v_i to v_{i+1}, and $\mathcal{P}_3 = (w_1, w_2, \cdots, w_n)$, where (w_i, w_{i+1}) is directed from w_i to w_{i+1}, and (iii) edge (r, u_1) directed from r to u_1, edge (r, v_1) directed from v_1 to r, and edge (r, w_1) directed from w_1 to r; see Fig. 3.b. Let P be a convex point set with $3n + 1$ points such that a set Q of $(3n - 1)/2$ points $q_1, q_2, \cdots, q_{(3n-1)/2}$ are to the left and a set R of $(3n - 1)/2$ points $r_1, r_2, \cdots, r_{(3n-1)/2}$ are to the right of the line connecting $p_M(P)$ and $p_m(P)$. Assume that $y(p_m(P)) < y(q_1) < y(r_1) < y(q_2) < y(r_2) < \cdots < y(q_{(3n-1)/2}) < y(r_{(3n-1)/2}) < y(p_M(P))$; see Fig. 3.c. It is not difficult to show that T does not admit any upward straight-line embedding into P.

Theorem 5. *For every n odd greater or equal than 5, there exists a $(3n + 1)$-vertex directed tree T and a $(3n + 1)$-point convex point set P such that T does not admit a straight-line upward embedding into P.*

4 Upward Simultaneous Embeddings of Directed Graphs

First, we show two n-vertex upward planar directed graphs that do not admit any upward simultaneous embedding without mapping. Let G_n^1 be the graph with $n \geq 5$ vertices $u_1^1, u_2^1, \cdots, u_n^1$, with a Hamiltonian directed path $\mathcal{P}_1 = (u_1^1, u_2^1, \cdots, u_n^1)$, with edges (u_1^1, u_3^1), (u_1^1, u_4^1), (u_1^1, u_5^1), (u_2^1, u_4^1), and (u_2^1, u_5^1), and with any other set of edges such that G_n^1 is still upward planar. Let G_n^2 be the graph with n vertices $u_1^2, u_2^2, \cdots, u_n^2$, with a Hamiltonian directed path $\mathcal{P}_2 = (u_1^2, u_2^2, \cdots, u_n^2)$, with edges (u_1^2, u_3^2), (u_1^2, u_4^2), (u_1^2, u_5^2), (u_2^2, u_4^2), and (u_3^2, u_5^2), and with any other set of edges such that G_n^2 is still upward planar. It is not difficult to show that G_n^1 and G_n^2 do not admit any upward simultaneous embedding without mapping.

Theorem 6. *For every $n \geq 5$, there exist two n-vertex upward planar directed graphs that do not admit a simultaneous embedding without mapping.*

Next, we show that deciding whether two upward planar directed graphs G_1 and G_2 admit a simultaneous embedding without mapping is an $\mathcal{NP}$-hard problem (in the following called UPWARD SIMULTANEOUS EMBEDDING WITHOUT MAPPING). In order to prove the $\mathcal{NP}$-hardness of such a problem, we perform a reduction from 3-PARTITION [10]). An instance of 3-PARTITION consists of a set A of $3m$ elements, a bound $B \in Z^+$, and a size $s(a) \in Z^+$ for each element $a \in A$, such that $B/4 < s(a) < B/2$ and such that $\sum_{a \in A} s(a) = mB$. The 3-PARTITION problem is to decide whether A can be partitioned into m disjoint sets $A_1, A_2, \ldots, A_m$ such that, for $1 \leq i \leq 3$ and $1 \leq j \leq m$, $\sum_{a_i \in A_j} s(a_i) = B$.

We describe how to construct an instance of UPWARD SIMULTANEOUS EMBEDDING WITHOUT MAPPING from an instance of 3-PARTITION.

Graph G_1 (see Fig. 4.a) contains m directed paths $\mathcal{P}_i = (u_1^{i,1}, u_1^{i,2}, \ldots, u_1^{i,2B})$ of $2B$ vertices, with $1 \leq i \leq m$. Edge $(u_1^{i,j}, u_1^{i,j+1})$ is directed from $u_1^{i,j}$ to $u_1^{i,j+1}$, for $1 \leq j \leq 2B - 1$ and $1 \leq i \leq m$. Further, G_1 has an edge directed from vertex $u_1^{i,2B}$ to vertex $u_1^{i+1,2B}$, for each i odd such that $1 \leq i \leq m - 1$, and an edge directed from vertex $u_1^{i+1,1}$ to vertex $u_1^{i,1}$, for each i even such that $2 \leq i \leq m-1$. Finally, G_1 has two vertices w_1 and z_1 such that, for every vertex $u_1^{i,j}$, with $1 \leq i \leq m$ and $1 \leq j \leq 2B$,

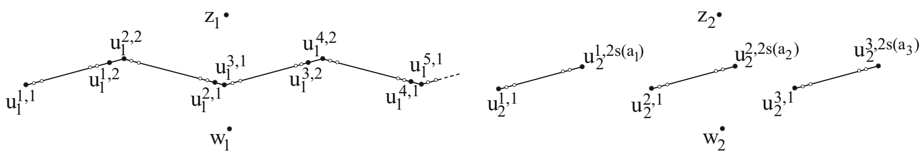


Fig. 4. (a)-(b) Graphs G_1 and G_2. In order to improve readability, edges incident to vertices w_1, z_1, w_2, and z_2 are not shown. Labels are shown only for the first and the last vertex of paths $\mathcal{P}_i$ of G_1 and for the first and the last vertex of the paths of each component G_2^i. An edge (a, b) is oriented from a to b if, in the figure, the y-coordinate of b is greater than the one of a.

there exists an edge from $u_1^{i,j}$ to z_1 and an edge from w_1 to $u_1^{i,j}$. It is easy to see that G_1 has only one upward planar embedding, up to a flip of the whole graph. Further, the subgraph $\mathcal{P}$ of G_1 induced by the all the vertices of G_1 except for w_1 and z_1 is a directed path. We say that two vertices $u_1^{i_1,j_1}$ and $u_1^{i_2,j_2}$ of G_1 are *consecutive* in G_1 if they are adjacent in $\mathcal{P}$.

Graph G_2 (see Fig. 4.b) has a triconnected component G_2^i for each element $a_i \in A$ (except for the elements a_i such that $s(a_i) = 1$ to which biconnected components G_2^i correspond). All the G_2^i's share two vertices w_2 and z_2. Graph G_2^i has $2 \cdot s(a_i) + 2$ vertices, where $2 \cdot s(a_i)$ vertices form a directed path $(u_2^{i,1}, u_2^{i,2}, \cdots, u_2^{i,2 \cdot s(a_i)})$ such that edge $(u_2^{i,j}, u_2^{i,j+1})$ is directed from $u_2^{i,j}$ to $u_2^{i,j+1}$, for $1 \leq j \leq 2 \cdot s(a_i) - 1$. For every $1 \leq i \leq 3m$ and $1 \leq j \leq 2 \cdot s(a_i)$, there is an edge directed from $u_2^{i,j}$ to z_2 and an edge directed from w_2 to $u_2^{i,j}$. Notice that, an embedding of G_2 is completely specified by a left-to-right order of the G_2^i's, up to flip of some G_2^i's. We say that two vertices $u_2^{i_1,j_1}$ and $u_2^{i_2,j_2}$ of G_2 are *consecutive* in G_2 if $i_1 = i_2$ and $j_2 = j_1 \pm 1$. Notice that, in any upward simultaneous embedding of G_1 and G_2, vertex w_1 of G_1 must be mapped to vertex w_2 of G_2 and vertex z_1 of G_1 must be mapped to vertex z_2 of G_2. We observe the following:

Lemma 7. *Let $u_2^{i,j}$ and $u_2^{i,j+1}$ be two consecutive vertices of G_2, for some $1 \leq i \leq 3m$ and $1 \leq j \leq 2 \cdot s(a_i) - 1$. In any upward simultaneous embedding without mapping of G_1 and G_2, vertices $u_2^{i,j}$ and $u_2^{i,j+1}$ are mapped to consecutive vertices of G_1.*

Corollary 1. *Consider any upward simultaneous embedding without mapping of G_1 and G_2 in which two vertices $u_1^{i_1,j_1}$ and $u_1^{i_2,j_2}$ of G_1 have been mapped to two vertices of the same component G_2^k of G_2. All vertices between $u_1^{i_1,j_1}$ and $u_1^{i_2,j_2}$ in $\mathcal{P}$ have been mapped to vertices of G_2^k.*

Lemma 8. *In any upward simultaneous embedding without mapping of G_1 and G_2 there exists no component G_2^i of G_2 which has two vertices mapped to vertices $u_1^{j,2B}$ and $u_1^{j+1,2B-1}$, for every j odd, and there exists no component G_2^i of G_2 which has two vertices mapped to vertices $u_1^{j,1}$ and $u_1^{j+1,2}$, for every j even.*

We obtain the following:

Theorem 7. *UPWARD SIMULTANEOUS EMBEDDING WITHOUT MAPPING is $\mathcal{NP}$-hard.*

Proof. The reduction described at the beginning of the section can clearly be performed in polynomial time. In fact, 3-PARTITION is $\mathcal{NP}$-hard in the strong sense, hence it is $\mathcal{NP}$-hard even if $2mB$, which is the size of the constructed instance of UPWARD SIMULTANEOUS EMBEDDING WITHOUT MAPPING, is bounded by a polynomial in m. We show that an instance of 3-PARTITION admits a solution if and only if the corresponding instance of UPWARD SIMULTANEOUS EMBEDDING WITHOUT MAPPING admits a solution.

Consider any instance A of 3-PARTITION admitting a solution $A_1, A_2, \dots, A_m$ such that, for $1 \leq i \leq 3$ and $1 \leq j \leq m$, $\sum_{a_i \in A_j} s(a_i) = B$. We show how to map the vertices of G_2 to the vertices of G_1. For each $i = 1, \cdots, m$, consider components G_2^x, G_2^y, and G_2^z of G_2 corresponding to elements a_x, a_y, and a_z of A_i, respectively. Since G_2^k has $2 \cdot s(a_k)$ vertices different from w_2 and z_2, then G_2^x, G_2^y, and G_2^z have exactly $2B$ vertices different from w_2 and z_2. Map vertex $u_2^{x,j}$ of G_2^x to vertex $u_1^{i,j}$ of G_1, for each $1 \leq j \leq 2 \cdot s(a_x)$; map vertex $u_2^{y,j}$ of G_2^y to vertex $u_1^{i,2 \cdot s(a_x)+j}$ of G_1, for each $1 \leq j \leq 2 \cdot s(a_y)$; map vertex $u_2^{z,j}$ of G_2^z to vertex $u_1^{i,2 \cdot s(a_x)+2 \cdot s(a_y)+j}$ of G_1, for each $1 \leq j \leq 2 \cdot s(a_z)$. It is easy to see that G_2, when its vertices are mapped to the vertices of G_1 as described above, is a subgraph of G_1. Hence, an upward simultaneous embedding of G_1 and G_2 is obtained by any upward straight-line drawing of G_1.

Now consider any upward simultaneous embedding (Γ_1, Γ_2) of G_1 and G_2. We show how to construct a solution $A_1, A_2, \dots, A_m$ for the corresponding instance A of 3-PARTITION. We claim that the vertices of three components $G_2^{x,i}$, $G_2^{y,i}$, and $G_2^{z,i}$ of G_2, except for w_2 and z_2, have been mapped to all and only the vertices of path $\mathcal{P}_i$ of G_1, for each $i = 1, 2, \cdots, m$. The claim directly implies the existence of a solution to the instance of 3-PARTITION, since the claim implies that $|G_2^{x,i}| + |G_2^{y,i}| + |G_2^{z,i}| = 2B$ and hence, by construction, the three elements of A corresponding to $G_2^{x,i}$, $G_2^{y,i}$, and $G_2^{z,i}$ sum up to B. Consider the component $G_2^{x,1}$ of G_2 which has a vertex mapped to $u_1^{1,1}$. By Corollary 1, the vertices of $G_2^{x,1}$ are mapped to consecutive vertices of G_1, hence they are mapped to the vertices of $\mathcal{P}$ from $u_1^{1,1}$ to $u_1^{1,|G_2^{x,1}|}$. Analogously, the component $G_2^{y,1}$ of G_2 which has a vertex mapped to $u_1^{1,|G_2^{x,1}|+1}$ has vertices mapped to vertices of $\mathcal{P}$ from $u_1^{1,|G_2^{x,1}|+1}$ to $u_1^{1,|G_2^{x,1}|+|G_2^{y,1}|}$. Observe that $|G_2^{x,1}| + |G_2^{y,1}| < 2B$, since each component of G_2 has less than B vertices (by the assumption that $s(a_i) < B/2$). Hence, there exists a component $G_2^{z,1}$ of G_2 which has a vertex mapped to $u_1^{1,|G_2^{x,1}|+|G_2^{y,1}|+1}$. By Corollary 1, the vertices of $G_2^{z,1}$ are mapped to consecutive vertices of G_1. Suppose that $|G_2^{x,1}| + |G_2^{y,1}| + |G_2^{z,1}| > 2B$. Since each of $|G_2^{x,1}|$, $|G_2^{y,1}|$, and $|G_2^{z,1}|$ is even, $|G_2^{x,1}| + |G_2^{y,1}| + |G_2^{z,1}|$ is even, as well, hence $G_2^{z,1}$ has a vertex mapped to $u_1^{2,2B-1}$. By Lemma 8, (Γ_1, Γ_2) is not an upward simultaneous embedding without mapping of G_1 and G_2. Now suppose that $|G_2^{x,1}| + |G_2^{y,1}| + |G_2^{z,1}| < 2B$. Then, there exists a component $G_2^{w,1}$ that is mapped to $u_1^{1,|G_2^{x,1}|+|G_2^{y,1}|+|G_2^{z,1}|+1}$. By Corollary 1, the vertices of $G_2^{w,1}$ are mapped to consecutive vertices of G_1. Further, by construction and by the assumption that $s(a_i) > B/4$, $|G_2^{x,1}| + |G_2^{y,1}| + |G_2^{z,1}| + |G_2^{w,1}| > 2B$. Hence, since $|G_2^{x,1}| + |G_2^{y,1}| + |G_2^{z,1}| + |G_2^{w,1}|$ is even, $G_2^{w,1}$ has vertices mapped both to $u_1^{1,2B}$ and to $u_1^{2,2B-1}$. By Lemma 8, (Γ_1, Γ_2) is not an upward simultaneous embedding without mapping of G_1 and G_2. It follows that $|G_2^{x,1}| + |G_2^{y,1}| + |G_2^{z,1}| = 2B$. The previous argument can be iterated to show that the vertices of three components $G_2^{x,i}$, $G_2^{y,i}$, and $G_2^{z,i}$ of G_2, except for w_2 and z_2, have been mapped to all and only the vertices of path $\mathcal{P}_i$ of G_1, for each $i = 1, 2, \cdots, m$, proving the claim and hence the theorem. □

5 Conclusions and Open Problems

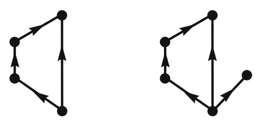
In this paper we have shown combinatorial and complexity results regarding the problem of constructing upward straight-line embeddings of directed graphs into point sets.

We have shown families of directed graphs that admit a straight-line upward embedding into every point set in convex or in general position, and families that do not. However, the problem of characterizing those graphs admitting a straight-line upward embedding into every point set in general or in convex position is still open. With this in mind we note that if an upward planar directed graph admits an upward straight-line embedding into every point set in general or in convex position, not all its subgraphs, in general, admit an upward straight-line embedding into every point set in general or in convex position; see Fig. 5. However, the necessary conditions of Lemmata 2, 3, and 4 strongly restrict the class of upward planar directed graphs to investigate. The following two problems naturally arise from the results of this paper: (1) Does every directed path admit a straight-line upward embedding into every point set in general position? (2) Does every directed caterpillar admit a straight-line upward embedding into every point set in convex/general position?

Deciding whether a directed graph admits an upward straight-line embedding into a given point set in general position is likely to be $\mathcal{NP}$-hard, since the same problem is $\mathcal{NP}$-hard in its undirected version [4]. However, we do not know the time complexity of testing whether a directed graph admits an upward straight-line embedding into a given point set in convex position. The same problem can be solved in linear time for undirected graphs as a graph admits a straight-line embedding into a given point set in convex position if and only if it is outerplanar, which can be tested in linear time [16].

We proved that deciding whether two upward planar directed graphs admit an upward simultaneous embedding without mapping is $\mathcal{NP}$-hard. We observe that the same problem is polynomial-time solvable for trees [11]. Hence, it would be interesting to solve the problem for subclasses of planar digraphs richer than directed trees, e.g., *outerplanar digraphs* and *series-parallel digraphs*.

A final open problem is to find the minimum cardinality $f(n)$ of a set of points P in the plane such that every n-vertex planar digraph admits an upward straight-line



(a) (b)

Fig. 5. (a) An upward planar directed graph G_1 that, by Theorem 1, does not admit an upward straight-line embedding into every 4-point point set in convex position. (b) An upward planar directed graph G_2 that admits an upward straight-line embedding into every 5-point point set in convex position and that contains G_1 as a subgraph.

embedding in which the vertices are drawn at points of P. While this is the directed version of one of the most studied Graph Drawing problems [5,6,14,15], the only known result is that the minimum size of any grid into which every planar digraph can be drawn is exponential [8], hence a polynomial upper bound for $f(n)$ would be interesting.

References

1. Bose, P.: On embedding an outer-planar graph in a point set. Computat. Geom. Th. Appl. 23(3), 303–312 (2002)
2. Bose, P., McAllister, M., Snoeyink, J.: Optimal algorithms to embed trees in a point set. J. Graph Algorithms Appl. 1(2), 1–15 (1997)
3. Braß, P., Cenek, E., Duncan, C.A., Efrat, A., Erten, C., Ismailescu, D., Kobourov, S.G., Lubiw, A., Mitchell, J.S.B.: On simultaneous planar graph embeddings. Computat. Geom. Th. Appl. 36(2), 117–130 (2007)
4. Cabello, S.: Planar embeddability of the vertices of a graph using a fixed point set is NP-hard. J. Graph Algorithms Appl. 10(2), 353–366 (2006)
5. Chrobak, M., Karloff, H.: A lower bound on the size of universal sets for planar graphs. Sigact News 20(4), 83–86 (1989)
6. de Fraysseix, H., Pach, J., Pollack, R.: How to draw a planar graph on a grid. Combinatorica 10(1), 41–51 (1990)
7. Di Battista, G., Tamassia, R.: Algorithms for plane representations of acyclic digraphs. Theor. Comput. Sci. 61, 175–198 (1988)
8. Di Battista, G., Tamassia, R., Tollis, I.G.: Area requirement and symmetry display of planar upward drawings. Disc. & Computat. Geometry 7, 381–401 (1992)
9. Estrella-Balderrama, A., Frati, F., Kobourov, S.: Upward straight-line embeddings of directed graphs into point sets. Tech. Report RT-DIA-133-2007, University of Roma Tre (2008), `http://dipartimento.dia.uniroma3.it/ricerca/rapporti/rt/2008-133.pdf`
10. Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-Completeness. W.H. Freeman, New York (1979)
11. Giordano, F., Liotta, G., Mchedlidze, T., Symvonis, A.: Computing upward topological book embeddings of upward planar digraphs. In: Tokuyama, T. (ed.) ISAAC 2007. LNCS, vol. 4835, pp. 172–183. Springer, Heidelberg (2007)
12. Gritzmann, P., Mohar, B., Pach, J., Pollack, R.: Embedding a planar triangulation with vertices at specified positions. The American Mathematical Monthly 98, 165–166 (1991)
13. Kaufmann, M., Wiese, R.: Embedding vertices at points: Few bends suffice for planar graphs. J. Graph Algorithms Appl. 6(1), 115–129 (2002)
14. Kurowski, M.: A 1.235 lower bound on the number of points needed to draw all -vertex planar graphs. Inf. Process. Lett. 92(2), 95–98 (2004)
15. Schnyder, W.: Embedding planar graphs on the grid. In: SODA 1990, pp. 138–148 (1990)
16. Wiegers, M.: Recognizing outerplanar graphs in linear time. In: Tinhofer, G., Schmidt, G. (eds.) WG 1986. LNCS, vol. 246, pp. 165–176. Springer, Heidelberg (1987)