Characterizations of Restricted Pairs of Planar Graphs Allowing Simultaneous Embedding with Fixed Edges

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Abstract. A set of planar graphs share a *simultaneous embedding* if they can be drawn on the same vertex set V in the Euclidean plane without crossings between edges of the same graph. *Fixed edges* are common edges between graphs that share the same simple curve in the simultaneous drawing. Determining in polynomial time which pairs of graphs share a simultaneous embedding *with* fixed edges (SEFE) has been open.

We give a necessary and sufficient condition for when a pair of graphs whose union is homeomorphic to K_5 or $K_{3,3}$ can have an SEFE. This allows us to determine which (outer)planar graphs *always* an SEFE with any other (outer)planar graphs. In both cases, we provide efficient algorithms to compute the simultaneous drawings. Finally, we provide an linear-time decision algorithm for deciding whether a pair of biconnected outerplanar graphs has an SEFE.

1 Introduction

In many practical applications including the visualization of large graphs and very-large-scale integration (VLSI) of circuits on the same chip, edge crossings are undesirable. A single vertex set can be used with multiple edge sets that each correspond to different edge colors or circuit layers. While the pairwise union of all edge sets may be nonplanar, a planar drawing of each layer may be possible, as crossings between edges of distinct edge sets are permitted. This is the problem of *simultaneous embedding* (SE) that generalizes the notion of planarity among multiple graphs.

Without restrictions on the types of edges, any number of planar graphs can be drawn on the same fixed set of vertex locations [16]. However, difficulties arise once straight-line edges are required. Moving one vertex to reduce crossings in one layer can introduce additional crossings in other layers. This is the problem of *simultaneous geometric embedding* (SGE). If edge bends are allowed, then having common edges drawn the same way using the same simple curve preserves the "mental map". Such edges are *fixed edges* leading to the problem of *simultaneous embedding with fixed edges* (SEFE). Since straight-line edges between a pair of vertices are also fixed, any graph that has an SGE also has an SEFE, but the converse is not true; see Fig. 1 that shows SGE $\subset$ SEFE $\subset$ SE.

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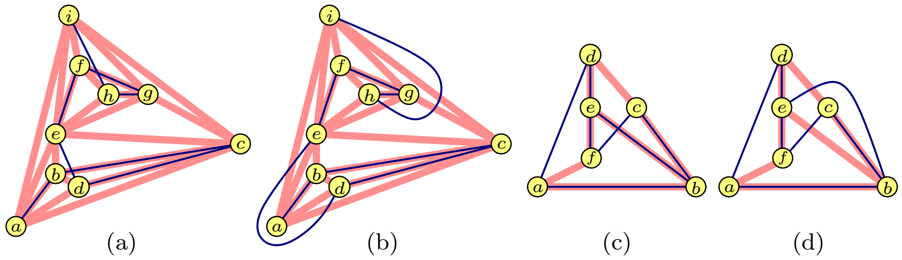


Fig. 1. The path and planar graph in (a) do not have an SGE with straight-line edges [2], but have an SEFE in (b). The two outerplanar graphs in (c) do not have an SEFE, but have an SE in (d) if edge (b, e) is not fixed.

Deciding whether two graphs have an SGE is NP-hard [9], whereas, deciding whether three graphs have an SEFE is NP-complete [13]. However, deciding whether two graphs have an SEFE in polynomial-time remains open. We give a necessary condition in terms of forbidden minors for when pairs of graphs can have an SEFE. While this does not yet lead to a polynomial-time decision algorithm in the general case, it does in the more restricted case of pairs of biconnected outerplanar graphs. Additionally, we determine which planar graphs and which outerplanar graphs always have an SEFE with any planar or outerplanar graph, respectively. We provide simultaneous drawings when possible.

1.1 Related Work

Any number of stars, two caterpillars (trees whose removal of all leaves gives a path) and two cycles always have an SGE, whereas three paths [2] and two trees [14] may not. Which trees and which graphs always have an SGE with a path, a caterpillar, a tree, or a cycle remains unknown.

The closest that any of these questions have been answered is for *unlabeled level planar* (ULP) graphs. A graph has an SGE with any path drawn in a strictly y-monotone fashion if and only if the graph is ULP [8]. ULP trees and graphs were recently determined and characterized in terms of two forbidden trees and five other forbidden graphs [7, 11]. If $O(1)$ bends per edge are allowed, three bends per edge suffice for pairs of planar graphs [6], while one bend per edge suffices for an outerplanar graph and a straight-line path [4].

For the case of SEFE, a planar graph and a tree can always be done, whereas two outerplanar graphs cannot [12]. This shows that the topological problem of SEFE is less restricted than the geometric problem of SGE. This is unlike standard planarity in which the sets of topological and geometric planar graphs are identical [3]. Planar graphs are characterized in terms of the forbidden graphs, K_5 and $K_{3,3}$ [15, 17]. These form two minimum examples of nonplanarity. No similar description for SEFE, even for restricted pairs of planar graphs, in terms minimal forbidden pairs has been given until now.

A related problem is finding the *thickness* of a graph G, which is the minimum number of planar subgraphs whose union is G. If vertices are co-located and straight-line edges are used as in SGE, the minimum number of subgraphs is the *geometric thickness* of G. Using simultaneous embedding techniques, it was shown that graphs with degree at most 4 have geometric thickness 2 [5].

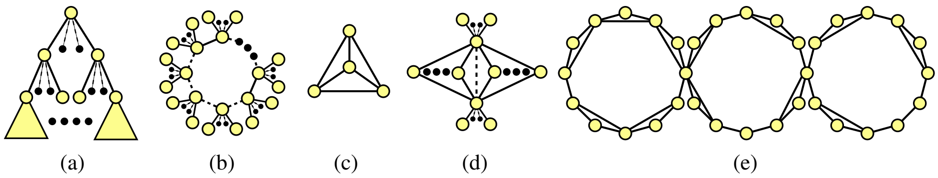


Fig. 2. Forests in (a), circular caterpillars in (b), K_4 in (c) and subgraphs of K_3-multiedges in (d) have an SEFE with any planar graph. Outerplanar graphs whose biconnected components form K_3-cycles in (e) have an SEFE with any outerplanar graph.

1.2 Our Contribution

1. We show there exist three paths without an SEFE. We provide a necessary and sufficient condition in terms of 17 minimal forbidden pairs for when a pair of graphs whose union forms a subdivided K_5 or $K_{3,3}$ has an SEFE.
2. Using this condition we show that the only graphs that *always* have an SEFE with *any* planar graph are either (i) forests, (ii) circular caterpillars (removal of all degree-1 vertices yields a cycle), (iii) K_4, or (iv) subgraphs of K_3-multiedges (any number of K_3's that all share an edge); see Fig. 2(a)–(d).
3. We show that this condition also implies that the only outerplanar graphs that *always* share an SEFE with *any* outerplanar graph are ones in which for each chord (x, y) there is a vertex z whose incident edges are (x, z) and (y, z). The biconnected components of these outerplanar graphs are called K_3-*cycles*; see Fig. 2(e). For each, we provide efficient drawing algorithms.
4. Using a forbidden outerplanar pair as a necessary and sufficient condition, we give a linear-time decision algorithm for deciding if two biconnected outerplanar graphs have an SEFE. Table 1 summarizes our results.

	SGE		SEFE				
	Path	Tree	Forest	Circular caterpillar	K_4 subgraph	K_3-multi-edge	biconnected K_3-cycles
Path	✓ [2]	?	✓ [12]	✓ [12]	✓ [12]	✓ [12]	✓ [12]
Caterpillar	✓ [2]	?	✓ [12]	✓ [12]	✓ [12]	✓ [12]	✓ [12]
Tree	?	✗ [14]	✓ [12]	✓ [12]	✓ [12]	✓ [12]	✓ [12]
Outerplanar	?	✗ [14]	✓	✓	✓	✓	✓
Planar	✗ [2]	✗ [2, 14]	✓	✓	✓	✓	✗

Table 1. Old and new results for SGE and SEFE pairs. The shaded pairs are new.

1.3 Preliminaries

Let P be a set of n distinct points in the xy-plane. A *planar drawing* of $G(V, E)$ consists of a bijection $\sigma : V \mapsto P$ with a simple curve for each edge $(u, v) \in E$ drawn in the xy-plane connecting the points $\sigma(u)$ and $\sigma(v)$ with curves only intersecting at endpoints. Let $\mathcal{G}$ be a set of planar graphs $\{G_1(V, E_1), G_2(V, E_2), \ldots, G_k(V, E_k)\}$. $\mathcal{G}$ has a *simultaneous embedding* if there exist planar drawings of $G_i(V, E_i)$ with the same bijection $\sigma : V \mapsto P$. If each edge is composed of one straight-line segment, then $\mathcal{G}$ has a *simultaneous geometric embedding* (SGE). If every common edge in $\mathcal{G}$ connecting a pair of vertices uses the same simple curve, then $\mathcal{G}$ has a *simultaneous embedding with fixed edges* (SEFE).

Two vertices u and v are *adjacent* if $(u, v) \in E$. A vertex u and edge (v, w) are *incident* if $u = v$ or $u = w$, and *nonincident*, otherwise. Likewise, two edges e and f are *incident* if they have a common endpoint. The *degree* of a vertex v, $deg(v)$, is the number of edges incident to v.

In a graph $G(V, E)$, *subdividing* an edge $(u, v) \in E$ replaces edge (u, v) with the pair of edges (u, w) and (w, v) in E by adding w to V. A *subdivision* of G is obtained through a series of edge subdivisions. *Contraction* of edge (u, v) replaces the vertices u and v with the vertex w that is adjacent to all the vertices that were adjacent to either u or v. A *minor* H of G is obtained through a series of edge contractions and deletions. A *pair* (G_1, G_2) consists of two graphs with the same set of vertices. A *minor pair* (H_1, H_2) of the (G_1, G_2) consists of the minors H_1 of G_1 and H_2 of G_2 each obtained by simultaneously contracting an edge in *both* graphs or deleting an edge from *either* graph. A graph $G(V, E)$ is *isomorphic* to a graph $\tilde{G}(\tilde{V}, \tilde{E})$ if there exists a bijection $f : V \mapsto \tilde{V}$ such that $(u, v) \in E$ if and only if $\big(f(u), f(v)\big) \in \tilde{E}$. A graph $G(V, E)$ is *homeomorphic* to a graph $\tilde{G}(\tilde{V}, \tilde{E})$ if a subdivisions of G and $\tilde{G}$ are isomorphic. The *induced subgraph* of G of the subset $V' \subseteq V$ is the subgraph given by the edge set $E \cap (V' \times V')$.

2 Forbidden Simultaneous Embeddings with Fixed Edges

We begin with Kuratowski's and Wagner's planar graph theorems [15, 17].

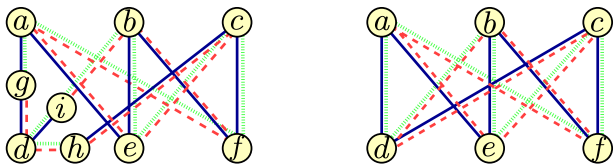
Theorem 1 (Kuratowski, Wagner) *A graph is nonplanar if and only if it has a subgraph homeomorphic or has a minor isomorphic to K_5 or $K_{3,3}$.*

2.1 Forbidden Triples of Paths and Cycles

Next we show the triples without an SGE of three paths [2] and three cycles [1] extend to SEFE.

Theorem 2 *There exist three paths on 9 vertices and three cycles on 6 vertices without an SEFE.*

Proof. Consider the three paths g–d–h–c–e–a–f–b–i, h–d–i–b–e–c–f–a–g, and i–d–g–a–e–b–f–c–h and the three cycles a–d–c–f–b–e–a, a–e–c–d–b–f–a, and a–f–c–e–b–d–a shown in Fig. 3. In both cases, the union forms a subdivided $K_{3,3}$ and must have a crossing by Theorem 1 in any drawing. Each edge in the union belongs to two paths (or cycles). Such a crossing must then be between two pairs of paths (or cycles). Since there are only three paths (or cycles) and fixed edges are being used, one path (or cycle) must self-intersect. □



(a) 3 paths on 9 vertices (b) 3 cycles on 6 vertices

Fig. 3. Two graph triples without an SEFE.

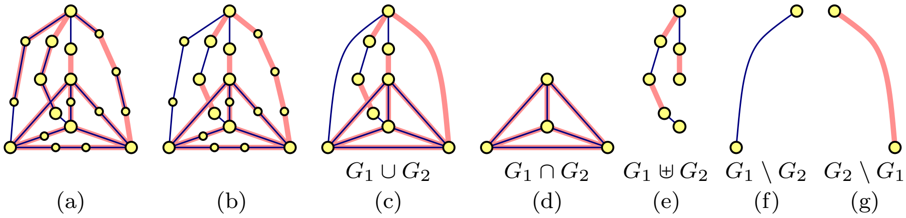


Fig. 4. Removing extraneous edges from (a) gives (b). Unsubdividing vertices gives (c) with four subgraphs, (d)–(g).

2.2 Minimal Forbidden Pairs

Suppose a pair of graphs $G_1(V, E_1)$ and $G_2(V, E_2)$ does not have an SEFE as in Fig. 4(a). If deleting any edge from either graph allows an SEFE, then G_1 and G_2 are *edge minimal* as in Fig. 4(b). If a degree-2 vertex v (adjacent to u and w) in the union is not a degree-1 vertex in either G_1 or G_2, then we can *unsubdivide* the vertex by deleting v and replacing edges (u, v) and (v, w) with the edge (u, w) in G_1 and/or G_2. A pair of graphs for which this can no longer be done is *vertex minimal* as in Fig. 4(c). A *minimal forbidden pair* does not have an SEFE and is edge and vertex minimal.

We define the *union* $G_1 \cup G_2$ and the *intersection* $G_1 \cap G_2$ with edge sets $E_1 \cup E_2$ and $E_1 \cap E_2$, respectively; see Fig. 4(c)–(d). Suppose then that $G_1 \cup G_2$ is homeomorphic to a graph G with no degree-2 vertices. Let $u \rightsquigarrow v$ in $G_1 \cup G_2$ be the path corresponding to the subdivided edge (u, v) in G. Path $u \rightsquigarrow v$ *is incident to* $x \rightsquigarrow y$ if and only if (u, v) and (x, y) are incident in G. An *alternating edge* is a $u \rightsquigarrow v$ path in which the edges strictly alternate between being in either G_1 and G_2, but not both; see Fig. 4(e). An *exclusive edge* is a $u \rightsquigarrow v$ path composed of the edge (u, v) that is only in G_1 or G_2; see Fig. 4(f)–(g), while an *inclusive edge* is composed of the fixed edge (u, v) in $G_1 \cap G_2$; see Fig. 4(c).

Claim 3 *Any pair of graphs $G_1(V, E_1)$ and $G_2(V, E_2)$ can be reduced to a pair in which every $u \rightsquigarrow v$ path is either an inclusive, exclusive, or alternating edge.*

Proof. We examine each $u \rightsquigarrow v$ path p in $G_1 \cup G_2$. If path p is in $G_1 \cap G_2$, we replace p with a single inclusive edge (u, v) in both G_1 and G_2. If p is in G_i but is missing edges in G_j for $i \neq j$, we replace it with the single exclusive edge (u, v) in G_i. If p is missing an edge from each graph, we make p into an alternating edge by deleting edges from p in either G_1 or G_2 until each edge along p is no longer in $G_1 \cap G_2$. Then we unsubdivide p until it is strictly alternating. We can always avoid crossings along edges of $u \rightsquigarrow v$ paths contained in $G_1 \cap G_2$ reduced in this way. Hence, neither operation changes whether the pair has an SEFE. □

Suppose G_1 and G_2 are a *reduced pair*, a pair of graphs in which all $u \rightsquigarrow v$ paths have been reduced. The *alternating edge subgraph*, $G_1 \uplus G_2$, is the subgraph of $G_1 \cup G_2$ consisting only of alternating edges. The *exclusive edge subgraph* of G_1, $G_1 \setminus G_2$, is the subgraph of $G_1 \cup G_2$, of exclusive edges from G_1, where $G_2 \setminus G_1$ is defined analogously. Hence, edges of $G_1 \cup G_2$ are partitioned into $G_1 \uplus G_2$, $G_1 \setminus G_2$, $G_2 \setminus G_1$, and $G_1 \cap G_2$; see Fig. 4(c)–(g). Next we see why we only need to consider crossings between nonincident edges.

Observation 4 *Crossings in a nonplanar drawing between incident edges can be removed without affecting the number of crossings of nonincident edges.*

This can be done by swapping the simple curves from the incident vertex to the first intersection point p. Separating the curves at p by a distance ε eliminates the crossing without affecting the rest of the drawing. Repeating this process removes all crossings of incident edges. Hence, we only need to consider crossings of nonincident edges in a simultaneous drawing with fixed edges. Applying this observation to the minimal examples of Theorem 1 gives the next corollary.

Corollary 5 *(a) Every drawing of K_5 or $K_{3,3}$ has a crossing between nonincident edges. (b) K_5 or $K_{3,3}$ can be drawn with only one crossing between any pair of nonincident edges.*

We use this corollary to produce a sufficient condition for SEFE.

Lemma 6 *Suppose the union of reduced pair G_1 and G_2 is homeomorphic to K_5 or $K_{3,3}$. Let $u \rightsquigarrow v$ and $x \rightsquigarrow y$ be nonincident paths in $G_1 \cup G_2$ but not in $G_1 \cap G_2$. G_1 and G_2 share an SEFE if either path belongs to $G_1 \uplus G_2$ or one belongs to $G_1 \setminus G_2$ and the other belongs to $G_2 \setminus G_1$.*

Proof. By Corollary 5(b), a K_5 or $K_{3,3}$ can always be drawn so that only (u,v) and (x,y) cross. Hence, there is an SEFE in which an alternating edge in $G_1 \uplus G_2$ only crosses an edge in either $G_1 \setminus G_2$ or $G_2 \setminus G_1$. Likewise, an edge in $G_1 \setminus G_2$ can cross any nonincident edge in $G_2 \setminus G_1$. □

With Lemma 6 we determine when a K_5 or a $K_{3,3}$ pair has an SEFE.

Corollary 7 *Suppose the union of reduced pair G_1 and G_2 is homeomorphic to K_5 or $K_{3,3}$. G_1 and G_2 have no SEFE if and only if (i) every nonincident edge of an alternating edge in $G_1 \uplus G_2$ is in $G_1 \cap G_2$ and (ii) every nonincident edge of an exclusive edge in $G_1 \setminus G_2$ is in G_1.*

Proof. For necessity, suppose G_1 and G_2 do not have an SEFE. If there is a nonincident edge $x \rightsquigarrow y$ of an alternating edge $u \rightsquigarrow v$ that is not in $G_1 \cap G_2$, by Lemma 6, G_1 and G_2 would have an SEFE since $u \rightsquigarrow v$ is in $G_1 \uplus G_2$ and neither path is in $G_1 \cap G_2$. If there is a nonincident edge $x \rightsquigarrow y$ of an edge $(u,v) \in G_1 \setminus G_2$ that is not in G_1, then again by Lemma 6, G_1 and G_2 would have an SEFE since $x \rightsquigarrow y$ is either in $G_1 \uplus G_2$ or $G_2 \setminus G_1$.

For sufficiency, suppose conditions (i) and (ii) hold. Since the union forms a subdivided K_5 or $K_{3,3}$, by Corollary 5(a) at least one pair of nonincident paths $u \rightsquigarrow v$ and $x \rightsquigarrow y$ cross. If either is in $G_1 \cap G_2$, then there must be a crossing in G_1 or G_2. If either is in $G_1 \uplus G_2$, then by (i) the other would be in $G_1 \cap G_2$, again giving a crossing in G_1 or G_2. If both are in $G_i \setminus G_j$ for $i \neq j$, then there is a crossing in G_i. Finally, (ii) prevents one edge being in $G_1 \setminus G_2$ and the other edge being in $G_2 \setminus G_1$. Hence, G_1 and G_2 do not have an SEFE. □

Theorem 8 *There are 17 minimal forbidden pairs with a union homeomorphic to K_5 or $K_{3,3}$.*

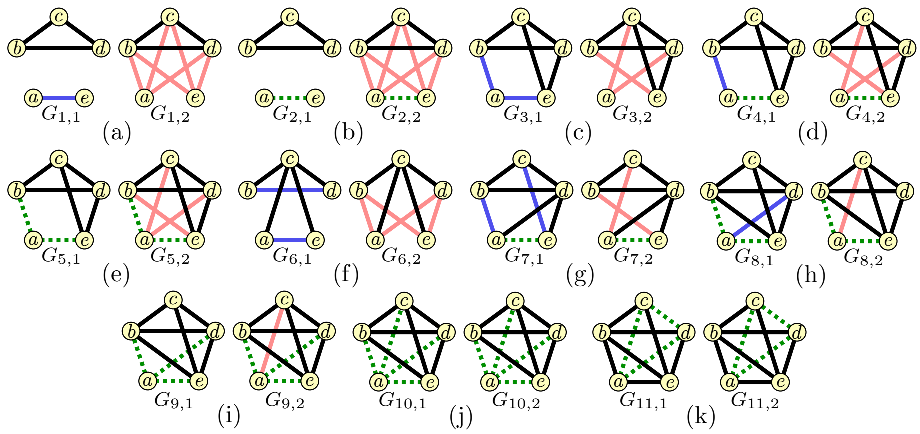


Fig. 5. Eleven K_5 minimal forbidden pairs.

Proof. Let $G_{i,j}$ denote the 17 pairs of graphs for $i \in \{1, \ldots, 17\}$ and $j \in \{1, 2\}$ in Figs. 5 and 6. One can verify that all the nonincident edges of any alternating edge are in the intersection and every edge nonincident to an exclusive edge of $G_{i,1}$ is also in $G_{i,1}$. This satisfies Corollary 7 implying that none of these pairs has an SEFE. Removing any edge means either (i) the union no longer forms a K_5 or a $K_{3,3}$ or (ii) the intersection does not contain all the nonincident edges of $G_{i,1} \uplus G_{i,2}$ or of $G_{i,1} \setminus G_{i,2}$ (other than those already in $G_{i,1}$) so that Corollary 7 is no longer satisfied. This implies that all 17 forbidden pairs are minimal.

We next show that our set of 17 pairs are the only minimal forbidden pairs homeomorphic to K_5 or $K_{3,3}$. Assume w.l.o.g. (G_1, G_2) are a reduced minimal forbidden pair whose union forms a K_5 or $K_{3,3}$ where G_2 has at least as many edges as G_1. We consider all the possibilities for edges to be in $G_1 \setminus G_2$ or $G_1 \uplus G_2$.

Pairs $(G_{1,1}, G_{1,2})$, $(G_{2,1}, G_{2,2})$, $(G_{12,1}, G_{12,2})$, and $(G_{13,1}, G_{13,2})$ are the only possibilities in which there is one exclusive edge in G_1 or one alternating edge in $G_1 \uplus G_2$. Two nonincident alternating edges would violate Corollary 7. The other case for a pair of nonincident edges are two exclusive edges in G_1 given by pairs $(G_{6,1}, G_{6,2})$ and $(G_{14,1}, G_{14,2})$. Three nonincident edges are only possible in a $K_{3,3}$, but including all nonincident edges implies G_1 is the whole $K_{3,3}$.

For the case of $G_1 \cup G_2$ homeomorphic to K_5, the $(G_{3,1}, G_{3,2})$, $(G_{4,1}, G_{4,2})$, and $(G_{5,1}, G_{5,2})$ pairs give the three distinct possibilities of two incident edges that are exclusive and/or alternating. Two incident exclusive edges with a third exclusive or alternating edge, incident or not, is not possible for the following reason: $G_{3,1}$ with two incident exclusive edges has seven edges. Adding another exclusive or alternating edge along with its nonincident edge, would leave only one edge for $G_2 \setminus G_1$, contradicting our assumption of $G_1 \setminus G_2$ being no larger.

Two nonincident exclusive edges with a third nonincident alternating edge is given by the pair $(G_{7,1}, G_{7,2})$. Three alternating edges that are all incident with another exclusive edge in G_2 or alternating edge in $G_1 \uplus G_2$ are given by pairs $(G_{9,1}, G_{9,2})$ and $(G_{10,1}, G_{10,2})$, respectively. The last possibility of three alternating edges that are only pairwise incident is given by pair $(G_{11,1}, G_{11,2})$ in which all the nonincident edges of each alternating edge is in the intersection.

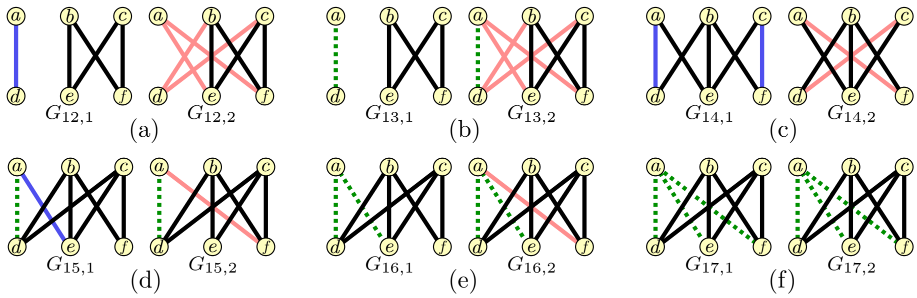


Fig. 6. Six $K_{3,3}$ minimal forbidden pairs.

For $G_1 \cup G_2$ homeomorphic to $K_{3,3}$, if there are two incident exclusive and/or alternating edges, then all remaining edges except for one must be in G_1. This edge $u \rightsquigarrow v$ is the one incident to both in the union. Edges nonincident to $u \rightsquigarrow v$ are also in G_1, so $G_2 \setminus G_1$ can only contain the edge $u \rightsquigarrow v$. Hence, $G_1 \setminus G_2$ has at most one edge by assumption. Pairs $(G_{15,1}, G_{15,2})$ with one exclusive edges and one alternating edge and $(G_{16,1}, G_{16,2})$ with two alternating edges are the only possibilities for two incident edges. Thus, a third edge can only be an alternating edge. However, $G_{16,2}$ already has one exclusive edge with two incident alternating edges leaving three incident alternating edges that are all incident given by pair $(G_{17,1}, G_{17,2})$ as the final possibility. □

Unlike a single planar graph that has the same forbidden minors as forbidden subdivisions by Theorem 1, the same is not true for SEFE. Fig. 7 shows three pairs with the same minor pair $(G_{7,1}, G_{7,2})$ in Fig. 7(a). Each pair is obtained by "uncontracting" vertex d to form the fixed edge (d_1, d_2) in Figs. 7(b)–(d). Fig. 7(b)–(c) are forbidden pairs, whereas, Fig. 7(d) is not.

Figs. 7(c)–(d) are examples in which a new fixed edge (a, d) is created from the exclusive edges (a, d_1) in $G_1 \setminus G_2$ and (a, d_2) in $G_2 \setminus G_1$ by contracting edge (d_1, d_2) to vertex d in Fig. 7(a). To avoid this, we define a *fixed edge minor pair* as a minor pair (H_1, H_2) of (G_1, G_2) that is obtained by only contracting edges in which no new fixed edges are created. Fig. 7(b) is an example in which Fig. 7(a) forms a fixed edge minor pair. This leads to following corollary.

Corollary 9 *Pair (G_1, G_2) has no SEFE if the pair has a fixed edge minor pair (H_1, H_2) isomorphic to one of the 17 minimal forbidden pairs of Theorem 8.*

This forms a necessary condition for SEFE, but is not a sufficient given that Fig. 7(c) without an SEFE does not have any of the 17 fixed edge minor pairs.

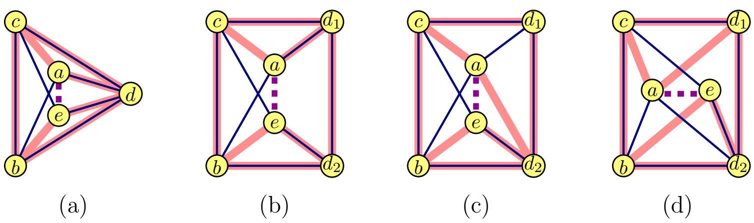


Fig. 7. $(G_{7,1}, G_{7,2})$ in (a) is a minor pair of the two forbidden pairs in (b) and (c) without an SEFE as well as the pair in (d) with the given SEFE.

3 Characterizing SEFE with Planar Graphs

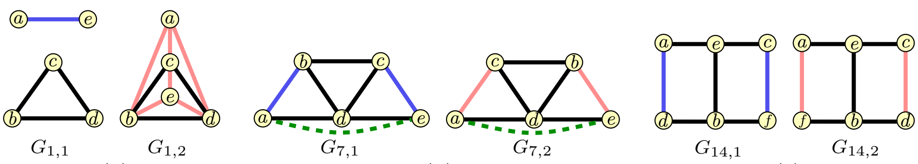
Next, we determine the graphs that *always* have an SEFE with *any* planar graph and produce simultaneous drawings. Let $\mathcal{P}$ be the set of planar graphs and $\mathcal{P}_{\mathsf{SEFE}}$ be the subset of $\mathcal{P}$ containing forests, *circular caterpillars* (removal of all degree-1 vertices leaves a cycle), K_4, and the subgraphs of K_3-*multiedges* (graphs with vertex set $\{x, y, z_1, \ldots, z_k\}$ and edge set $\{(x, y), (x, z_i), (y, z_i) : 1 \leq i \leq k\}$).

Lemma 10 *$\mathcal{P}_{\mathsf{SEFE}}$ are the only graphs that may always have an SEFE with any planar graph.*

Proof. Let $G_1 \in \mathcal{P}_{\mathsf{SEFE}}$ and $G_2 \in \mathcal{P}$. Both graphs of all the 17 pairs of Theorem 8 have a subgraph homeomorphic to $G_{1,1}$, a K_3 and a disjoint edge; see Fig. 8(a). First, we show that G_1 does not contain a subgraph homeomorphic to $G_{1,1}$. A forest has no cycles unlike $G_{1,1}$. While a circular caterpillar has a cycle, all other edges are incident to the cycle. A K_4 has four vertices while $G_{1,1}$ has five. Finally, every subgraph of a K_3-multiedge with a cycle, either has a 3-cycle $x \rightsquigarrow y \rightsquigarrow z \rightsquigarrow x$ or a 4-cycle $x \rightsquigarrow z_1 \rightsquigarrow y \rightsquigarrow z_2 \rightsquigarrow x$ (if there is no edge (x, y)). In either case, every other edge is part of the cycle or is incident to one of the cycle edges. This implies (G_1, G_2) cannot have an SEFE with G_2 by Corollary 9.

Next, we show that any graph $G \in \mathcal{P} \setminus \mathcal{P}_{\mathsf{SEFE}}$ contains a subgraph homeomorphic to $G_{1,1}$, preventing an SEFE with the set of all planar graphs that includes the graph $G_{1,2}$. The graph G must have a cycle since otherwise it would be a forest. Let C be a cycle in G with maximum length and e be any edge in $G \setminus C$. Clearly, e is incident to C or G would contain a subgraph homeomorphic to $G_{1,1}$. If the edge e forms a chord of C where C has length greater than four, then there a cycle C' formed by a path in C and the edge e that would have a nonincident edge e' in C such that C' and e' would be homeomorphic to $G_{1,1}$.

Hence, all cycles in G have length 4 or less. Assume C is a 3-cycle with another cycle C' in G. Either C and C' share an edge giving a longer cycle (contradicting the maximality of C) or C' would have an edge nonincident to C. Hence, C is a 4-cycle if G has multiple cycles. If two 4-cycles C and C' only share a vertex or an single edge, then C would have a nonincident edge in C'. Hence, C and C' must share two edges. If they are nonincident, then C_1 and C_2 forms a K_4. This implies that G either forms a K_4 or that all the 4-cycles share a common path consisting of the two incident edges (x, z) and (y, z) implying that all 3-cycles share the common edge (x, y), if it exists. Any non-cycle edge e must be incident to all the cycles implying that e is either (x, w) or (y, w) for some degree-1 vertex w. This implies that if G has multiple cycles, but is not a K_4, then it is a subgraph of some K_3-multiedge. Finally, if C is the only cycle, then all the vertices not in C have degree-1 giving a circular caterpillar. □



(a) (b) (c)

Fig. 8. Planar drawings for the pairs $(G_{1,1}, G_{1,2})$, $(G_{7,1}, G_{7,2})$, and $(G_{14,1}, G_{14,2})$.

Lemma 10 provides a necessary condition for the graphs that may have an SEFE with any planar graph. This condition also suffices by the following lemma whose omitted proof appears in [10]. The key idea is to compute a SEFE algorithm with Euclidean shortest paths to draw each edge not in the intersection. Such a path always exists since (i) a forest only has one face, (ii) a circular caterpillar only has one cycle with all other edges are incident, (iii) a K_4 is a 4-cycle C with two chords (with one drawn outside of C and the other inside) and (iv) a subgraph of K_3-multiedge that is not a circular caterpillar has a 4-cycle in which all other edges are incident to one of the two vertices x and y of the 4-cycle.

Lemma 11 *An SEFE exists for any graph in $\mathcal{P}_{\mathsf{SEFE}}$ with any planar graph.*

Lemmas 10 and 11 together imply the following theorem.

Theorem 12 *A graph G always has an SEFE with any planar graph if and only $G \in \mathcal{P}_{\mathsf{SEFE}}$.*

4 Characterizing SEFE with Outerplanar Graphs

We next determine which outerplanar graphs *always* have an SEFE with *any* other outerplanar graph. A K_3*-cycle* is a biconnected outerplanar graph such that for every chord (x, y), there is a vertex z whose incident edges are (x, z) and (y, z). Let $\mathcal{O}$ be the set of outerplanar graphs and $\mathcal{O}_{\mathsf{SEFE}} \subset \mathcal{O}$ be the set of outerplanar graphs in which each biconnected component is a K_3-cycle. Note that the condition on each vertex z (adjacent to endpoints of a chord (x, y)) still applies in that no edges from other components can be incident to z.

The following Lemma provides an analogous result for outerplanar graphs that Lemma 10 does for planar graphs. The omitted proof found in [10] uses an approach similar used to prove Lemma 10 in which $G_{1,1}$ served as forbidden graph. The set $\mathcal{O}_{\mathsf{SEFE}}$ is shown to be the set of outerplanar graphs that do not contain either $(G_{7,1}, G_{7,2})$ or $(G_{14,1}, G_{14,2})$ as a fixed edge minor pair, which are the only forbidden pairs from Theorem 8 in which both graphs are outerplanar.

Lemma 13 *$\mathcal{O}_{\mathsf{SEFE}}$ are the only outerplanar graphs that may always have an SEFE with any outerplanar graph.*

The condition of Lemma 13 also suffices for outerplanar graphs by the following lemma whose omitted also proof appears in [10]. Euclidean shortest paths are again used to draw each edge not in the intersection. The cycles along the outerface for each biconnected component in $\mathcal{O}_{\mathsf{SEFE}}$ are each completed (in depth first order) so as not to contain any other by routing the final cycle edge in a clockwise direction around the boundary of what has been drawn so far. Any remaining chords can always be drawn inside the outerface since each has a degree-2 vertex z on the outerface that is adjacent to both endpoints.

Lemma 14 *An SEFE exists for any graph in $\mathcal{O}_{\mathsf{SEFE}}$ with any outerplanar graph.*

Lemmas 13 and 14 together give the following theorem.

Theorem 15 *An outerplanar graph G always has an SEFE with any outerplanar graph if and only $G \in \mathcal{O}_{\mathsf{SEFE}}$.*

5 Deciding SEFE for Biconnected Outerplanar Graphs

While Corollary 9 is not sufficient in general, we can show sufficiency for the more restrictive case of pairs of biconnected outerplanar graphs.

Lemma 16 *The outerplanar pair* (G_1, G_2) *has an SEFE if and only if the pair does not have the fixed edge minor pair* $(G_{14,1}, G_{14,2})$.

The omitted proof found in [10] provides an algorithm to compute a SEFE in which all homeomorphic subgraphs of each outerplanar pair that match the outerplanar graphs of $(G_{14,1}, G_{14,2})$ (but do not match the forbidden labeling) are drawn using an approach similar to the drawing algorithm for Lemma 14 in which cycles are closed in such a way as not to prevent other cycles from being closed.

Theorem 17 *Deciding whether a pair of biconnected outerplanar graphs* (G_1, G_2) *on* n *vertices has an SEFE can be done in* $O(n)$ *time.*

The omitted proof found in [10] uses the conditions on each chord in the intersection given by the drawing algorithm in the proof of Lemma 16. This condition can be checked in constant time, which yields a linear-time decision algorithm.

6 Conclusion

We gave a necessary condition, which is not sufficient, for two graphs to have an SEFE in terms of 17 forbidden fixed edge minor pairs. This allows us to determine which (outer)planar graphs always have an SEFE with any (outer)planar graphs closing several open questions from Table 1.

We showed sufficiency for the restricted case of two biconnected outerplanar graphs that only has one forbidden minor pair, namely $(G_{14,1}, G_{14,2})$, allowing us to produce a polynomial time decision algorithm, the first polynomial-time decision algorithm (of which we are aware) for any SEFE pair.

Future work includes includes finding all fixed edge minor pairs of planar graphs, which would give a sufficient condition for their SEFE. This may lead to a polynomial time decision algorithm demonstrating a fundamental difference between the the topological problem of SEFE and geometric problem of SGE for which such a decision algorithm is NP-hard [9].

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