# Straight-line Grid Drawings of 3-Connected 1-Planar Graphs

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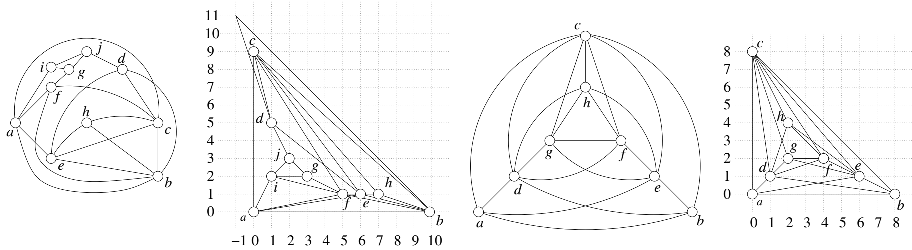
**Abstract.** A graph is 1-planar if it can be drawn in the plane such that each edge is crossed at most once. In general, 1-planar graphs do not admit straight-line drawings. We show that every 3-connected 1-planar graph has a straight-line drawing on an integer grid of quadratic size, with the exception of a single edge on the outer face that has one bend. The drawing can be computed in linear time from any given 1-planar embedding of the graph.

## 1 Introduction

Since Euler's Königsberg bridge problem dating back to 1736, planar graphs have provided interesting problems in theory and in practice. Fáry, Stein and Wagner proved independently that every planar graph has a straight-line planar drawing [16, 24, 28]. Using the elaborate techniques of a canonical ordering and Schnyder realizers, these results were improved to straight-line drawing on a grid of quadratic size, and such drawings can be computed in linear time [9, 23]. The area bound is asymptotically optimal, since the nested triangle graphs are planar graphs and require $\Omega(n^2)$ area [11]. The drawing algorithms were refined to improve the area requirement or to admit convex representations, i.e., where each inner face is convex [5, 8, 19] or strictly convex [1].

However, most graphs are nonplanar and recently, there have been many attempts to study larger classes of graphs. Of particular interest are 1-planar graphs, which in a sense are one step beyond planar graphs. These were introduced by Ringel [22] in an approach to color a planar graph and its dual. Although it is known that a 3-connected planar graph and its dual have a straight-line 1-planar drawing [27] and even on a grid of quadratic size [14], little is known about general 1-planar graphs. It is NP-hard to recognize 1-planar graphs [17, 20] in general, although there is a linear-time testing algorithm [12] for maximal 1-planar graphs (i.e., where no additional edge can be added without violating 1-planarity) given the the circular ordering of incident edges around each vertex. A 1-planar graph with $n$ vertices has at most $4n - 8$ edges [4, 15, 21] and this upper bound is tight. On the other hand straight-line drawings of 1-planar graphs may have at most $4n - 9$ edges and this bound is tight [10]. Hence not all 1-planar graphs admit straight-line drawings. Unlike planar graphs, maximal 1-planar graphs can be much sparser with only $2.64n$ edges [6].

Thomassen [26] refers to 1-planar graphs as graphs with *cross index 1* and proved that an embedded 1-planar graph can be turned into a straight-line drawing if and only if it excludes $B$- and $W$-configurations; see Fig. 2. These forbidden configurations were



(a) (b) (c) (d)

**Fig. 1.** (a)–(b) A 3-connected 1-planar graph and its straight-line grid drawing (with one bend in one edge), (c)–(d) another 3-connected 1-planar graph and its straight-line grid drawing.

first discovered by Eggleton [13] and used by Hong *et al.* [18], who show that the configurations can be detected in linear time if the embedding is given. They also proved that there is a linear time algorithm to convert a 1-planar embedding without $B$- and $W$-configurations into a straight-line drawing, but without bounds for the drawing area.

In this paper we settle the straight-line grid drawing problem for 3-connected 1-planar graphs. First we compute a *normal form* for an embedded 1-planar graphs with no $B$-configuration and at most one $W$-configuration on the outer face. Then, after augmenting the graph with as many planar edges as possible and then deleting the crossing edges, we find a 3-connected planar graph, which is drawn with strictly convex faces using an extension of the algorithm of Chrobak and Kant [8]. Finally the pairs of crossing edges are reinserted into the convex faces. This gives a straight-line drawing on a grid of quadratic size with the exception of a single edge on the outer face, which may need one bend (and this exception is unavoidable); see Fig. 1. In addition, the drawing is obtained in linear time from a given 1-planar embedding.

## 2 Preliminaries

We consider simple undirected graphs $G = (V, E)$ with $n$ vertices and $m$ edges. A *drawing* of a graph is a mapping of $G$ into the plane such that the vertices are mapped to distinct points and each edge is a Jordan arc between its endpoints. A drawing is *planar* if the Jordan arcs of the edges do not cross and it is *1-planar* if each edge is crossed at most once. Note that crossings between edges incident to the same vertex are not allowed. For example, $K_5$ and $K_6$ are 1-planar graphs. An *embedding* of a graph is planar (resp. 1-planar) if it admits a planar (resp. 1-planar) drawing. An embedding specifies the *faces*, which are topologically connected regions. The unbounded face is the *outer face*. A face in a planar graph is specified by a cyclic sequence of edges on its boundary (or equivalently by the cyclic sequence of the endpoints of the edges).

Accordingly, a 1-*planar embedding* $\mathcal{E}(G)$ specifies the faces in a 1-planar drawing of $G$ including the outer face. A 1-planar embedding is a witness for 1-planarity. In particular, $\mathcal{E}(G)$ describes the pairs of crossing edges and the faces where the edges cross. Each pair of *crossing edges* $(a, c)$ and $(b, c)$ induces a *crossing point* $p$. Call the segment of an edge between the vertex and the crossing point a *half-edge*. Each half-edge is *impermeable*, analogous to the edges in planar drawings, in the sense that no edge can cross such a half-edge without violating the 1-planarity of the embedding. The non-crossed edges are called *planar*. A *planarization* $G^{\times}$ is obtained from $\mathcal{E}(G)$

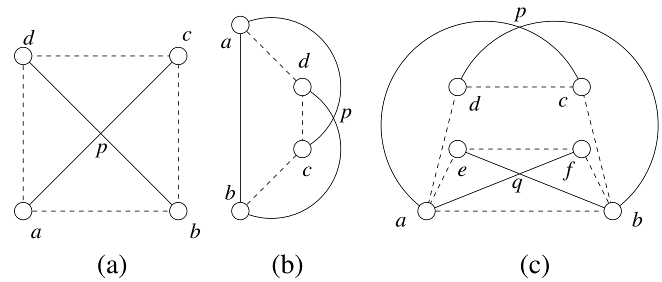



**Fig. 2.** (a) An augmented $X$-configuration, (b) an augmented $B$-configuration, (c) an augmented $W$-configuration. The graphs induced by the solid edges are called an $X$-configuration (a), a $B$-configuration (b), and a $W$-configuration (c).

by using the crossing points as regular vertices and replacing each crossing edge by its two half-edges. A 1-planar embedding $\mathcal{E}(G)$ and its planarization share equivalent embeddings, and each face is given by a list of edges and half-edges defining it, or equivalently, by a list of vertices and crossing points of the edges and half edges.

Eggleton [13] raised the problem of recognizing 1-planar graphs with rectilinear drawings. He solved this problem for outer-1-planar graphs (1-planar graphs with all vertices on the outer-cycle) and proposed three forbidden configurations. Thomassen [26] solved Eggleton's problem and characterized the rectilinear 1-planar embeddings by the exclusion of $B$- and *W-configurations*; see Fig. 2. Hong *et al.* [18], obtain a similar characterization where the $B$- and $W$-configurations are called the "Bulgari" and "Gucci" graphs. They also show that all occurrences of these configurations can be computed in linear time from a given 1-planar embedding.

**Definition 1.** *Consider a 1-planar embedding $\mathcal{E}(G)$:*

*A $B$-*configuration *consists of an edge $(a, b)$ and two edges $(a, c)$ and $(b, d)$ which cross in some point $p$ such that $c$ and $d$ lie in the interior of the triangle $(a, b, p)$. Here $(a, b)$ is called the* base *of the configuration.*

*An $X$-*configuration *consists of a pair $(a, c)$ and $(b, d)$ of crossing edges which does not form a $B$-configuration.*

*A* W-configuration *consists of two pairs of edges $(a, c)$, $(b, d)$ and $(a, e)$, $(b, f)$ which cross in points $p$ and $q$, such that $c, d, e, f$ lie in the interior of the quadrangle $a, p, b, q$. Here again the edge $(a, b, )$, if present is the base.*

*Observe that for all these configurations the base edges may be crossed by another edge, whereas the crossing edges are impermeable; see Fig 2.*

Thomassen [26] and Hong *et al.* [18] proved that for a 1-planar embedding to admit straight-line drawing, $B$- and $W$-configurations must be excluded:

**Proposition 1.** *A 1-planar embedding $\mathcal{E}(G)$ admits a straight-line drawing with a topologically equivalent embedding if and only if it does not contain a $B$- or a $W$-configuration.*

Augment a given 1-planar embedding $\mathcal{E}(G)$ by adding as many edges to $\mathcal{E}(G)$ as possible so that $G$ remains a simple graph and the newly added edges are planar in $\mathcal{E}(G)$. We call such an embedding a *planar-maximal* embedding of $G$ and the operation *planar-maximal augmentation*. (Note that Hong *et al.* [18] color the planar edges of a 1-planar embedding as red and call a planar-maximal augmentation a *red augmentation*.) The *planar skeleton* $\mathcal{P}(\mathcal{E}(G)))$ consists of the planar edges of a planar-maximal augmentation. It is a planar embedded graph, since all pairs of crossing edges are omitted. Note that the planar augmentation and the planar skeleton are defined for an embedding, not for a graph. A graph may have different embeddings which give raise to different configurations and augmentations. The notion of planar-maximal embedding is different from the notions of maximal 1-planar embeddings and maximal 1-planar graphs, which are such that the addition of any edge violates 1-planarity (or simplicity) [6].

The following claim, proven in many earlier papers [6, 15, 18, 25, 26], shows that a crossing pair of edges induces a $K_4$ in planar-maximal embedding, since missing edges of a $K_4$ can be added without inducing new crossings.

**Lemma 1.** *Let $\mathcal{E}(G)$ be a planar-maximal 1-planar embedding of a graph $G$ and let $(a, c)$ and $(b, d)$ be two crossing edges. Then the four vertices $\{a, b, c, d\}$ induce a $K_4$.*

By Lemma 1, for a planar-maximal embedding each $X$-, $B$, and $W$-configuration is augmented by additional edges. Here we define these augmented configurations.

**Definition 2.** *Let $\mathcal{E}(G)$ be a planar-maximal 1-planar embedding of a graph $G$. An* augmented $X$-configuration *consists of a $K_4$ with vertices $(a, b, c, d)$ such that the edges $(a, c)$ and $(b, d)$ cross inside the quadrangle $abcd$. An* augmented $B$-configuration *consists of a $K_4$ with vertices $(a, b, c, d)$ such that the edges $(a, c)$ and $(b, d)$ cross beyond the boundary of the quadrangle $abcd$. An* augmented W-configuration *consists of two $K_4$'s $(a, b, c, d)$ and $(a, b, e, f)$ one of which is in an augmented $X$-configuration and the other in an augmented $B$-configuration.*

*For an augmented $X$- or augmented $B$-configuration, the edges not inducing a crossing with other edges in the configuration defines a cycle, we call it the* skeleton*. In each configuration, the edges on the outer-boundary of the embedded configuration and not inducing a crossing with other edges in the configuration are the* base edges*.*

Using the results of Thomassen [26] and Hong *et al.* [18], we can now characterize when a planar-maximal 1-planar embedding of a graph admits a straight-line drawing:

**Lemma 2.** *Let $\mathcal{E}(G)$ be a planar-maximal 1-planar embedding of a graph $G$. Then there is a straight-line 1-planar drawing of $G$ with a topologically equivalent embedding as in $\mathcal{E}(G)$ if and only if $\mathcal{E}(G)$ does not contain an augmented $B$-configuration.*

*Proof.* Assume that $\mathcal{E}(G)$ contains an augmented $B$-configuration. Then it must contain a $B$-configuration and has no straight-line 1-planar drawing due to Proposition 1. Conversely, if $\mathcal{E}(G)$ has no straight-line 1-planar drawing then by it contains at least one $B$- or $W$-configuration. Since $\Gamma$ is a planar-maximal embedding, by Lemma 1 each crossing edge pair in $\mathcal{E}(G)$ induces a $K_4$. Thus the dotted edges in Fig. 2(b)–(c) must be present in any $B$- or $W$- configuration, inducing an augmented $B$-configuration. □

The *normal form* for an embedded 1-planar graph $\mathcal{E}(G)$ is obtained by first adding the four planar edges to form a $K_4$ for each pair of crossing edge while routing them closely to the crossing edges and then removing old duplicate edges if necessary. Such an embedding of a 1-planar graph is a normal embedding of it. A *normal planar-maximal augmentation* for an embedded 1-planar graph is obtained by first finding a normal form of the embedding and then by a planar-maximal augmentation.

**Lemma 3.** *Given a 1-planar embedding $\mathcal{E}(G)$, the normal planar-maximal augmentation of $\mathcal{E}(G)$ can be computed in linear time.*

*Proof.* First augment each crossing of two edges $(a, c)$ and $(b, d)$ to a $K_4$, such that the edges $(a, b), (b, c), (c, d), (d, a)$ are added and in case of a duplicate the former edge is removed. Then all augmented X-configurations are empty and contain no vertices inside their skeletons. Next triangulate all faces which do not contain a half-edge, a crossing edge, or a crossing point. Each step can be done in linear time. □

## 3 Characterization of 3-Connected 1-Planar Graphs

Here we characterize 3-connected 1-planar graphs by a normal embedding, where the crossings are augmented to $K_4$'s such that the resulting augmented X-configurations have vertex-empty skeletons and there is no augmented $B$-configuration except for at most one augmented W-configuration with a pair of crossing edges in the outer face.

Let $\mathcal{E}(G)$ be a 1-planar embedding of a graph $G$. Each pair of crossing edges induces a crossing point and the crossing edges and their half-edges are *impermeable* as they cannot be crossed by other edges without violating 1-planarity. An *impermeable path* in $\mathcal{E}(G)$ is an internally-disjoint sequence $P = v_1, p_1, v_2, p_2, \ldots, v_n, p_n, v_{n+1}$, where $v_1, v_2, \ldots, v_{n+1}$ are (regular) vertices of $G$, $p_1, p_2, \ldots, p_n$ are crossing points in $\mathcal{E}(G)$ and $(v_i, p_i)$, $(p_i, v_{i+1})$ for each $i \in \{1, 2, \ldots, n\}$ are half edges. If $v_{n+1} = v_0$, then $P$ is an *impermeable cycle*. An impermeable cycle is *separating* when it has vertices both inside and outside of it, since deleting its vertices disconnects $G$.

**Lemma 4.** *Let $G = (V, E)$ be a 3-connected 1-planar graph with a planar-maximal 1-planar embedding $\mathcal{E}(G)$. Then the following conditions hold.*

*A. (i) Two augmented B-configurations or two augmented X-configurations cannot be on the same side of a common base edge.*

*(ii) Suppose an augmented B-configuration $B$ and an augmented X-configuration $X$ are on the same side of a common base edge $(a, b)$. Let $p$ and $q$ be the crossing points for $X$ and $B$, respectively and let $R(X)$ and $R(B)$ be the regions inside the skeletons of $X$ and $B$. Then all vertices of $V \setminus \{a, b\}$ are inside the impermeable cycle $apbq$ if $R(X) \subset R(B)$; otherwise all vertices of $V \setminus \{a, b\}$ are outside the impermeable cycle $apbq$.*

*B. (i) If two augmented B-configurations are on opposite sides of a common base edge $(a, b)$, with crossing points $p$ and $q$, respectively, then all the vertices of $V \setminus \{a, b\}$ are inside the impermeable cycle $apbq$.*

*(ii) If two augmented X-configurations are on opposite sides of a common base edge $(a, b)$, with crossing points $p$ and $q$, respectively, then all the vertices of $V \setminus \{a, b\}$ are outside the impermeable cycle $apbq$.*

*(iii) An augmented B-configuration and an augmented X-configuration cannot share a common base edge from opposite sides.*

*Proof.* Condition A.(i) and B.(iii) hold because each of these configurations induces a separating impermeable $apbq$ cycle in $\mathcal{E}(G)$ with only two (regular) vertices from $G$, a

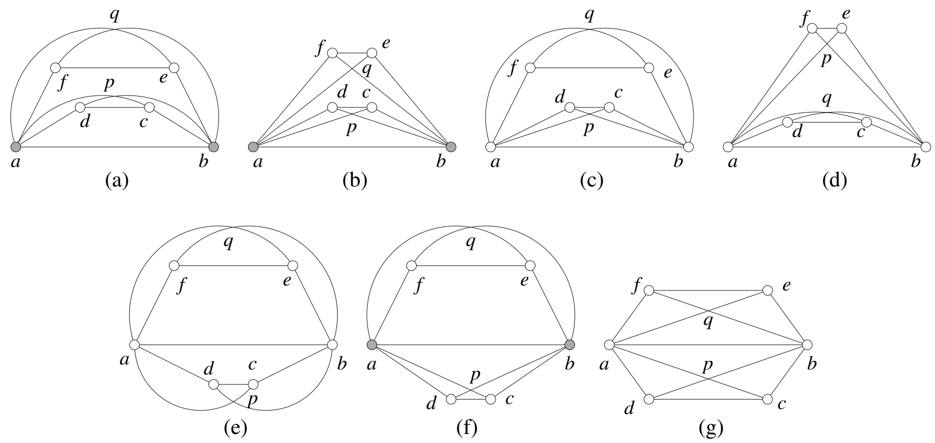



**Fig. 3.** Illustration for the proof of Lemma 4.

contradiction with the 3-connectivity of $G$; see Fig. 3(a)–(b) and (f). Similarly, if any of the Conditions A.(ii) and B.(i)–(ii) is not satisfied, then the impermeable cycle $apbq$ becomes separating and hence the pair $\{a, b\}$ becomes separation pair of $G$, again a contradiction with the 3-connectivity of $G$; see Fig. 3(c)–(d), (e) and (g). □

**Corollary 1.** *Let $G$ be a 3-connected 1-planar graph with a planar-maximal 1-planar embedding $\mathcal{E}(G)$. Then no three crossing edge-pairs in $\mathcal{E}(G)$ share the same base edge.*

*Proof.* Each crossing edge pair induces either an augmented B- or an augmented X-configuration. This fact along with Lemma 4[A.(i), B.(iii)] yields the corollary. □

**Lemma 5.** *Let $G$ be a 3-connected 1-planar graph. Then there is a planar-maximal 1-planar embedding $\mathcal{E}(G^*)$ of a super-graph $G^*$ of $G$ such that $\mathcal{E}(G^*)$ contains at most one augmented W-configuration and no other augmented $B$-configuration, and each augmented X-configuration in $\mathcal{E}(G^*)$ contains no vertex inside its skeleton.*

*Proof.* Let $\mathcal{E}(G)$ be a 1-planar embedding of $G$. We claim that by a normal planar-maximal augmentation of $\mathcal{E}(G)$ we get the desired embedding of a supergraph of $G$. Note that due to the edge-rerouting this operation converts any $B$-configuration whose base is not shared with another configuration into an $X$-configuration; see Fig. 4(a). On the other hand if a base edge is shared by two $B$-configurations, they are converted into one $W$-configuration and by Lemma 4 this $W$-configuration must be on the outerface; see Fig. 4(b). By Corollary 1, a base edge cannot be shared by more than two augmented $B$ configurations. Furthermore this operation does not create any new $B$-configurations. It also makes the skeleton of any augmented $X$-configuration vertex-empty, since by Lemma 4 the same base edge can be shared by at most two augmented $X$-configurations from the opposite side and in case it is shared by two augmented $X$-configuration, the interior of the induced impermeable cycle is empty; see Fig. 4(c). □

Lemma 5 together with Proposition 1 implies the following:

**Theorem 1.** *A 3-connected 1-planar graph admits a straight-line 1-planar drawing except for at most one edge in the outerface.*

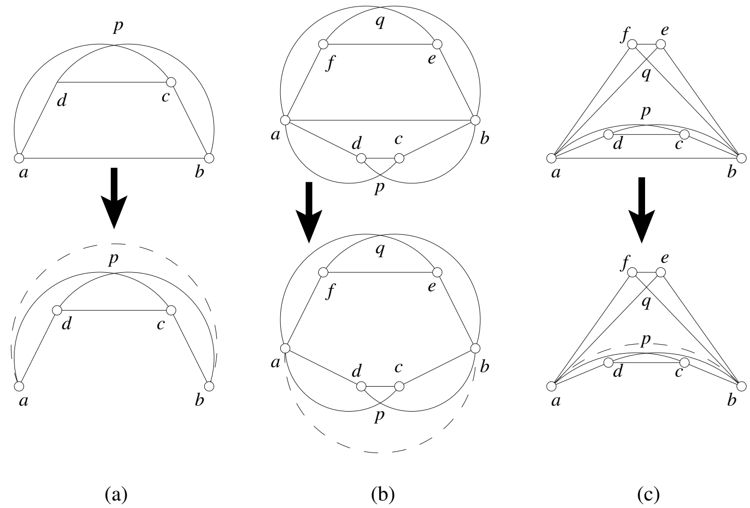



**Fig. 4.** Illustration for the proof of Lemma 5.

## 4 Grid Drawings

In the previous section we showed that a 3-connected 1-planar graph has a straight-line 1-planar drawing, with the exception of a single edge in the outer face, which comes from an unavoidable W-configuration. We now strengthen this result and show that there is straight-line grid drawing with $O(n^2)$ area, which can be constructed in linear time from a given 1-planar embedding.

The algorithm takes an embedding $\mathcal{E}(G)$ and computes a normal planar-maximal augmentation. Consider the planar skeleton $\mathcal{P}(\mathcal{E}(G))$ for the embedding. If there is an augmented W-configuration and a crossing in the outer face, one crossing edge on the outer face is kept and the other crossing edge is treated separately. Thus the outer face of $\mathcal{P}(\mathcal{E}((G))$ is a triangle and the inner faces are triangles or quadrangles. Each quadrangle comes from an augmented X-configuration. It must be drawn strictly convex, such that the crossing edges can be re-inserted. This is achieved by an extension of the convex grid drawing algorithm of Chrobak and Kant [8], which itself is an extension of the shifting method of de Fraysseix, Pach and Pollack [9]. Since the faces are at most quadrangles, we can avoid three collinear vertices and the degeneration to a triangle by an extra unit shift. Note that our algorithm achieves $O(n^2)$ area, while the general algorithms for strictly convex grid drawings [1, 7] require larger area, since strictly convex drawings of $n$-gons need $\Omega(n^3)$ area [2].

The algorithm of Chrobak and Kant and in particular the computation of a canonical decomposition presumes a 3-connected planar graph. Thus the planar skeleton of a 3-connected 1-planar graph must be 3-connected, which holds except for the $K_4$, when it is embedded as an augmented X-configuration. This results parallels the fact that the planarization of a 3-connected 1-planar graph is 3-connected [15].

**Lemma 6.** *Let $G$ be a graph with a planar-maximal 1-planar embedding $\mathcal{E}(G)$ such that it has no augmented B-configuration and each augmented X-configuration in $\mathcal{E}(G)$ has no vertex inside its skeleton. Then the planar skeleton $\mathcal{P}(\mathcal{E}(G))$ is 3-connected.*

We will prove Lemma 6 by showing that there is no separation pair in $\mathcal{P}(\mathcal{E}(G))$. First we obtain a planar graph $H$ from $G$ as follows. Let $(a, c)$ and $(b, d)$ be a pair of crossing edges that form an augmented X-configuration $X$ in $\Gamma$. We then delete the two edges $(a, c)$, $(b, d)$; add a vertex $u$ and the edges $(a, u)$, $(b, u)$, $(c, u)$, $(d, u)$ to triangulate the face $abcd$. Call $v$ a *cross-vertex* and call this operation *cross-vertex insertion* on $X$. We then obtain $H$ from $G$ by cross-vertex insertion on each augmented X-configuration. Call $H$ a *planarization* of $G$ and denote the set of all the cross-vertices by $U$. Then $\mathcal{P}(\mathcal{E}(G)) = H \setminus U$. Before proving Lemma 6 we consider several properties of $H$, the planarization of the 1-planar graph.

**Lemma 7.** *Let $G = (V, E)$ be a graph with a planar-maximal 1-planar embedding $\mathcal{E}(G)$ such that $\mathcal{E}(G)$ contains no augmented B-configuration and each augmented X-configuration in $\mathcal{E}(G)$ contains no vertex inside its skeleton. Let $H$ be a planarization of G, where $U$ is the set of cross-vertices. Then the following conditions hold.*

*(a) $H$ is a maximal planar graph (except if H is the $K_4$ in an X-configuration)*

*(b) Each vertex of $U$ has degree 4.*

*(c) $U$ is an independent set of $H$.*

*(d) There is no separating triangle of $H$ containing any vertex from $U$.*

*(e) There is no separating 4-cycle of $H$ containing two vertices from $U$.*

*Proof.* For convenience, we call each vertex in $V - U$ a *regular vertex*.

(a) Since $H$ is a planar graph, by definition we only need to show that each face of $H$ is a triangle. Each crossing edge pair in $\Gamma$ induces an augmented X-configuration whose skeleton has no vertex in its interior. Therefore each face of $H$ containing a crossing vertex is a triangle. On the other hand, Hong *et al.* [18] showed that in any planar-maximal 1-planar embedding a face containing no crossing vertices is a triangle. Thus $H$ is a maximal planar graph.

(b)–(c) These two conditions follow from the fact that the neighborhood of each crossing vertex consists of exactly four regular vertices that form the skeleton of the corresponding augmented X-configuration.

(d) For a contradiction suppose a vertex $u \in U$ participates in a separating triangle $T$ of $H$. Since the neighborhood of $u$ forms the skeleton of the corresponding augmented X-configuration $X$, the other two vertices, say $a$ and $b$, in $T$ are regular vertices. The edge $(a, b)$ cannot form a base edge for $X$, since if it did, then the interior of the separating triangle $T$ would be contained in the interior of the skeleton for $X$ and hence would be empty. Assume therefore that $a$ and $b$ are not consecutive on the skeleton of $X$. In this case the edge $(a, b)$ is a crossing edge in $G$ and hence has been deleted when constructing $H$; see Fig. 5(a).

(e) Suppose two vertices $u, v \in U$ participate in a separating 4-cycle of $H$. Due to Condition (c), assume without loss of generality that the separating 4-cycle is $T = aubv$, where $a$, $b$ are regular vertices. Assume first that the two vertices $a, b$ are adjacent in $H$ and also assume without loss of generality that the edge $(a, b)$ is drawn inside the interior of $T$. This means that the interior of at least one of the two triangles $abu$ and $abv$ is non-empty and hence at least one of these two triangles forms a separating triangle in $H$, a contradiction with Condition (d). We

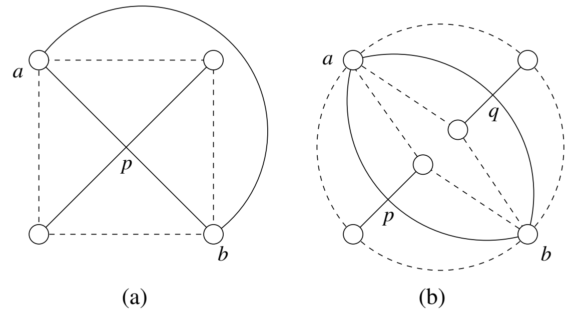



**Fig. 5.** Illustration for the proof of Lemma 6.

thus assume that the two vertices $a$ and $b$ are not adjacent in $H$. Then for both the augmented X-configurations $X$ and $Y$, corresponding to the two crossing vertices $u$ and $v$, the two vertices $u$ and $v$ are not consecutive on their skeleton. This implies that the crossing edge $(a, b)$ participates in two different augmented X-configurations in $\Gamma$, again a contradiction; see Fig. 5(b).

□

We are now ready to prove Lemma 6

*Proof (Lemma 6).* Assume for the purpose of obtaining a contradiction that $\mathcal{P}(\mathcal{E}(G))$ is not 3-connected. Then there exists some separation pair $\{a, b\}$ in $\mathcal{P}(\mathcal{E}(G))$. Let $H$ be the planarization of $G$, where $U$ is the set of cross-vertices. Then $S = U \cup \{a, b\}$ is a separating set for $H$. Take a minimal separating set $S' \subset S$ such that no proper subset of $S'$ is a separating set for $H$. Since $H$ is a maximal planar graph (from Lemma 7(a)), $S'$ must form a separating cycle [3]. As $H$ is a maximal planar graph it must be 3-connected, which implies that $|S'| \geq 3$. On the other hand, since $S'$ contains at most two regular vertices $a, b$ and no two cross-vertices can be adjacent in $H$ (Lemma 7(c)), $|S'| < 5$. Hence $S'$ is either a separating triangle or a separating 4-cycle in $H$ containing at most two regular vertices; we get a contradiction with Lemma 7(d)–(e). □

Finally, we describe our algorithm for straight-line grid drawings. This drawing algorithm is based on an extension of the algorithm of Chrobak and Kant [8] for computing a convex drawing of a planar 3-connected graph. For convenience we refer to this algorithm as the CK-algorithm and we begin with a brief overview. Let $G = (V, E)$ be an embedded 3-connected graph and let $(u, v)$ be an edge on the outer-cycle of $G$. The CK-algorithm starts by computing a *canonical decomposition* of $G$, which is an ordered partition $V_1, V_2, \ldots, V_t$ of V such that the following conditions hold:

(i) For each $k \in \{1, 2, \ldots, t\}$, the graph $G_k$ induced by the vertices $V_1 \cup \ldots \cup V_k$ is 2-connected and its outer-cycle $C_k$ contains the edge $(u, v)$.
(ii) $G_1$ is a cycle, $V_t$ is a singleton $\{z\}$, where $z \notin \{u, v\}$ is on the outer-cycle of $G$.
(iii) For each $k \in \{2, \ldots, t-1\}$ the following conditions hold:
   - If $V_k$ is a singleton {z}, then $z$ is on the outerface of $G_{k-1}$ and has at least one neighbor in $G - G_k$.
   - If $V_k$ is a chain $\{z_1, \ldots, z_l\}$, each $z_i$ has at least one neighbor in $G - G_k$, $z_1$, $z_l$ have one neighbor each on $C_{k-1}$ and no other $z_i$ has neighbors on $G_{k-1}$.

For each $k \in \{1, 2, \ldots, t\}$, the vertices that belong to $V_k$ have rank $k$. We call a vertex of $G_k$ *saturated* if it has no neighbor in $G - G_k$. The CK-algorithm starts by drawing the edge $(u, v)$ with a horizontal line-segment of unit length. Then for $k = 1, 2, \ldots, t$, it incrementally completes the drawing of $G_k$. Let $C_{k-1} = \{(u = w_1, \ldots, w_p, \ldots, w_q, \ldots, w_r = v)\}$ with $1 \leq p < q \leq r$ such that $w_p$ and $w_q$ are the leftmost and the rightmost neighbor of vertices in $V_k$. Then the vertices of $V_k$ are placed above the vertices $w_p, \ldots, w_q$. Assume that $V_k = \{z_1, \ldots, z_l\}$. Then $z_1$ is placed on the vertical line containing $w_p$ if $w_p$ is saturated in $G_k$; otherwise it is placed on the vertical line one unit to the right of $w_p$. On the other hand, $z_l$ is placed on the negative diagonal line (i.e., with $-45°$ slope) containing $w_q$. If $v_k$ is a singleton then $z = z_1 = z_l$ is placed at the intersection of these two lines. Otherwise (after necessary shifting of $w_q$ and other vertices), the vertices $z_1, \ldots z_l$ are placed on consecutive vertical lines one unit apart from each other. In order to make sure that this shifting operation does not disturb planarity or convexity, each vertex $v$ is associated with an "under-set" $U(v)$ and whenever $v$ is shifted, all vertices in $U(v)$ are also shifted along with $v$. Thus the edges between vertices of any $U(v)$ are in a sense *rigid*.

**Theorem 2.** *Given a 1-planar embedding $\mathcal{E}(G)$ of a 3-connected graph $G$, a straight-line drawing on the $(2n-2) \times (2n-3)$ grid can be computed in linear time. Only one edge on the outer face may require one bend.*

*Proof.* Assume that $\mathcal{E}(G)$ is a normal planar-maximal embedding; otherwise we compute one by a normal planar-maximal augmentation in linear time by Lemma 3. Consider the planar skeleton $\mathcal{P}(\mathcal{E}(G))$. If there is no unavoidable W-configuration on the outerface of the maximal planar augmentation, then the outer-cycle of $\mathcal{P}(\mathcal{E}(G))$ is a triangle. Otherwise we add one of the crossing edges in the outer face to $\mathcal{P}(\mathcal{E}(G))$ to make the outer-cycle triangle. The other crossing edge is treated separately. By Lemma 6, $\mathcal{P}(\mathcal{E}(G))$ is 3-connected, its outer face is a triangle $(a, b, c)$ and the inner faces are triangles or quadrangles, where the latter result from augmented X-configurations and are in one-to-one correspondence to pairs of crossing edges.

We wish to obtain a planar straight-line grid drawing of $\mathcal{P}(\mathcal{E}(G))$ such that all quadrangles are strictly convex. Although the CK-algorithm draws any 3-connected planar graph of $n$ vertices on a grid of size $(n-1) \times (n-1)$ with convex faces, the faces are not necessarily strictly convex [8]. Hence we must modify the algorithm so that all quadrangles are strictly convex. Note that by the assignment of the under-sets, the CK-algorithm guarantees that once a face is drawn strictly convex, it would remain strictly convex after any subsequent shifting of vertices.

For $\mathcal{P}(\mathcal{E}(G))$ each $V_k$ is either a single vertex or a pair with an edge, since the faces are at most quadrangles. If $V_k$ is an edge $(z_1, z_2)$ then, by the definition of the canonical decomposition, exactly one quadrangle face $w_p z_1 z_2 w_q$ is formed and by construction this face is drawn convex. We thus assume that $V_k$ contains a single vertex, say $v$. Let $C_{k-1} = \{(u = w_1, \ldots, w_p, \ldots, w_q, \ldots, w_r = v)\}$ with $1 \leq p < q \leq r$ such that $w_p$ and $w_q$ are the leftmost and the rightmost neighbor of vertices in $V_k$. Then the new faces created by the insertion of $v$ are all drawn strictly convex unless there is some quadrangle $v w_{p'-1} w_{p'} w_{p'+1}$ where $p < p' < q$ and $w_{p'-1}$, $w_{p'}$, $w_{p'+1}$ are collinear in the drawing of $G_{k-1}$. Note that in this case the vertex $w_{p'}$ must be saturated in $G_{k-1}$.

This case may occur in the CK-algorithm only when the line containing $w_{p'-1}$, $w_{p'}$, $w_{p'+1}$ is either a vertical line or a negative diagonal (with $-45°$ slope). In the former case, $w_{p-1}$ should have also been saturated in $G_{k-1}$, which is not possible since $v$ is its neighbor. Thus it is sufficient to make sure that no saturated vertex of $G_k$ is in the negative diagonal of both its left and right neighbor on $C_k$. We do this by the following extension of the CK-algorithm.

Suppose that $v$ is placed above $w_q$ with slope $-45$ and $w_q$ was placed above its rightmost lower neighbor $w'_{q'}$ with slope $-45$, and there is the quadrangle $(v, w_q, w'_{q'}, u)$ for some vertex $u$ with higher rank (i.e., which will be placed later). Then shift $w'_{q'}$ by one extra unit to the right when $v$ or $u$ is placed. This implies a bend at $w_q$ and sets a strictly convex angle above $w_q$.

The CK-algorithm starts by placing the first two vertices one unit away and it requires a unit shift to the right for each following vertex. On the other hand, a 1-planar graph has at most $n-2$ pairs of crossing edges. Hence, there are $g \leq n-3$ augmented X-configurations, each of which induces a quadrangle in the planar skeleton. Thus the width and height are $n-1+g$, which is bounded by $2n-4$. The vertices $a, b, c$ of the outer triangle are placed at the grid points $(0,0), (0, n-1+g), (n-1+g, 0)$.

In case the original graph had an unavoidable $W$-configuration in the outerface, we need a post-processing phase to add to the drawing the extra edge $(b,d)$, which induces a crossing in the outer face with the edge $(a,c)$. Since $a$ is the leftmost lower neighbor of $d$ when $d$ is placed and $d$ is not saturated, $d$ is placed in the first column at $(1,j)$ for some $j < n-2+g$. Shift $b$ one unit to the right, insert a bend point at $(-1, n+g)$ just one diagonal unit left above $c$ and route the edge $(b,d)$ via the bend point. □

Figure 6 in the appendix illustrates the operation of the algorithm for computing a straight-line drawing of a 3-connected 1-planar graph on a $(2n-2) \times (2n-3)$ grid.

## 5 Conclusion and Future Work

We have shown that 3-connected 1-planar graphs can be embedded on the $O(n) \times O(n)$ integer grid, so that edges are drawn as straight-line segments (except for at most one edge on the outerface that requires a bend). Moreover, the algorithm is simple and runs in linear time. Some 1-planar embedded graphs may require exponential area; see Hong *et al.* [18]. As we have shown, this cannot happen with 3-connected 1-planar graphs. It is not clear whether there exist biconnected 1-planar graphs for which any straight-line 1-planar drawing requires exponential area. Recognition of 1-planar graphs is NP-hard [20]. How hard is the recognition of planar-maximal 1-planar graphs?

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# Appendix

## Illustration of the Algorithm for a 3-Connected 1-planar Graph

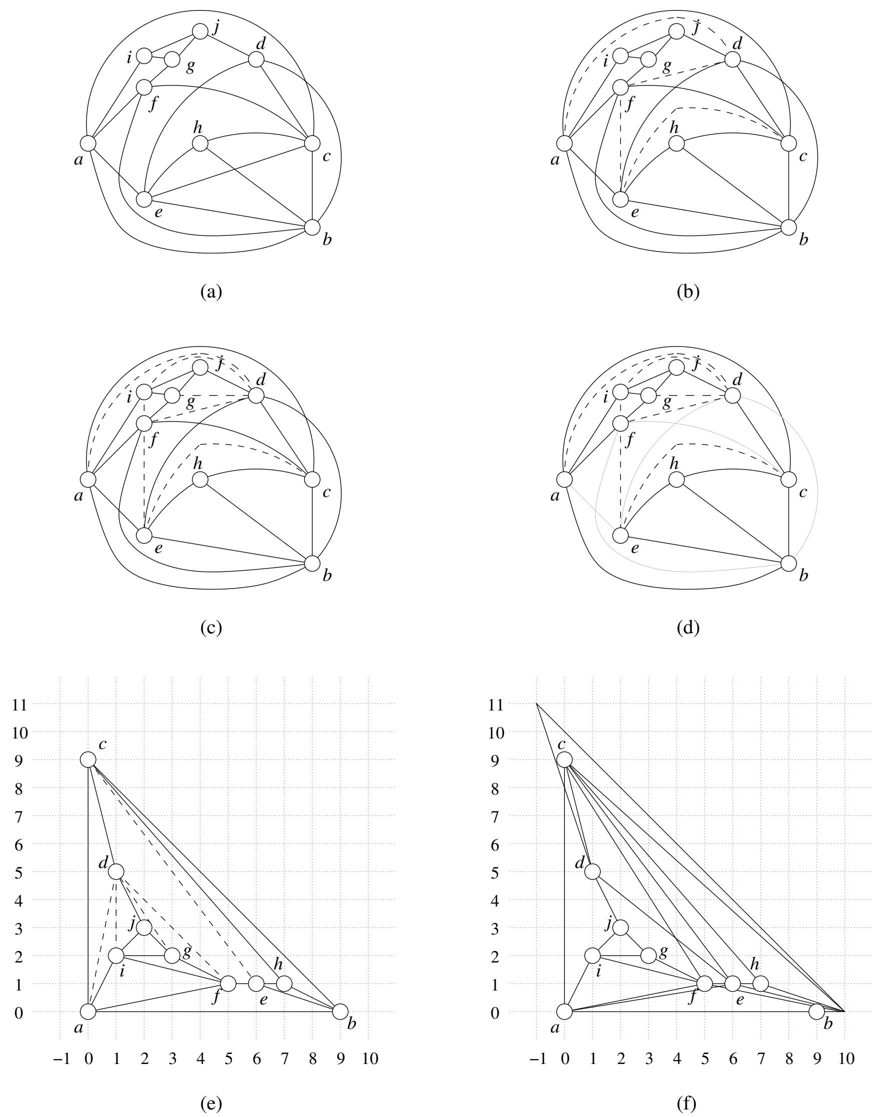


**Fig. 6.** (a) A 3-connected 1-planar graph $G$, (b) a normal embedding for $G$, (c) a planar-maximal normal embedding $\mathcal{E}^*(G)$, (d) a planar skeleton $\mathcal{P}(\mathcal{E}^*(G))$ computed from $\mathcal{E}^*(G)$ by deleting the crossing edges except one crossing edge on the outerface, (e) a straight-line strictly-convex grid drawing $\Gamma^*$ of $\mathcal{P}(\mathcal{E}^*(G))$ using an extension of the algorithm in [8], and (f) a grid drawing $\Gamma$ of $G$ with straight-line edges except for one edge with one bend.