Proportional Contact Representations of 4-connected Planar Graphs

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Abstract. In a contact representation of a planar graph, vertices are represented by interior-disjoint polygons and two polygons share a non-empty common boundary when the corresponding vertices are adjacent. In the weighted version, a weight is assigned to each vertex and a contact representation is called proportional if each polygon realizes an area proportional to the vertex weight. In this paper we study proportional contact representations of 4-connected internally triangulated planar graphs. The best known lower and upper bounds on the polygonal complexity for such graphs are 4 and 8, respectively. We narrow the gap between them by proving the existence of a representation with complexity 6. We then disprove a 10-year old conjecture on the existence of a Hamiltonian canonical cycle in a 4-connected maximal planar graph, which also implies that a previously suggested method for constructing proportional contact representations of complexity 6 for these graphs will not work. Finally we prove that it is **NP**-complete to decide whether a 4-connected planar graph admits a contact representation using only rectangles.

1 Introduction

Contact graph representations for planar graphs are a well-studied alternative to the traditional node-link diagram. In most contact graph representations, vertices are represented by geometric objects such as circles, triangles and rectangles, while edges correspond to two objects touching in some specified fashion.

Here we consider contact representations of planar graphs, with vertices represented by simple interior-disjoint polygons and adjacencies represented by non-trivial shared boundaries between the corresponding polygons. In the weighted version, the input is a planar graph $G = (V, E)$ along with a weight function $w : V \rightarrow \mathbb{R}^+$ that assigns a weight to each vertex of G. A *proportional contact representation* of G is a contact representation of G where each vertex v is represented by a polygon with $w(v)$ area. When the polygons are made using only axis-aligned sides, the unweighted and weighted contact representations are called *rectilinear duals* and *rectilinear cartograms*, respectively.

Contact representations have practical applications in cartography [14], geography [16], sociology [10] and floor-planning for VLSI layout [19]. Other applications are in visualization of relational data, where using the adjacency of regions to represent edges in a graph can lead to a more compelling visualization than just drawing a line segment between two points [3]. In this context it is often desirable, for aesthetic, practical and cognitive reasons, to limit the *polygonal complexity* of the representation, measured by the maximum number of sides in a polygon. Similarly, it is also desirable to minimize the unused area, also known as *holes* in floor-planning and VLSI layouts.

With these considerations in mind, we study the problem of constructing hole-free proportional contact representations with minimal polygonal complexity.

1.1 Related Work

Koebe's theorem [11] is an early example of a contact representation, showing that any planar graph can be represented by touching circles. Any planar graph also has a representation with triangles [5] and with cubes in 3D [8]. However all these results yield representations that contain point-contacts between adjacent polygons. When adjacent polygons must share nonempty common boundaries (also known as side-contact representation) it has been shown that convex hexagons are sometimes necessary and always sufficient [6]. The rectilinear variant of the side-contact representation problem was first studied in graph theoretic context, and then with renewed interest in the context of VLSI layouts and floor planning. It is known that 8 sides are sometimes necessary and always sufficient for rectilinear duals of maximal planar graphs [9, 13, 19]. Characterizations for planar graphs with rectilinear duals of complexity 4 and 6 are also known [12, 15, 17].

While all the above results deal with the unweighted version of the problem, the weighted version was first studied back in 1934 when Raisz described rectangular cartograms, i.e., rectilinear cartograms that use only rectangles [14]. In the general rectilinear setting it has been recently shown that 8 sides are sometimes necessary and always sufficient for a rectilinear cartogram of maximal planar graphs [2]. It has been shown that 7 sides are sometimes necessary and always sufficient if we drop the rectilinear restriction while still requiring proportional side-contact representations [1]. Note that the representation with 7 sides comes at the expense of many holes, when compared to the representation with 8 sides.

In this paper we study proportional contact representations for 4-connected internally-triangulated planar graphs. There are several earlier results on contact representation for this large graph class and for some subclasses thereof. It is known that the class of graphs that have rectilinear duals with rectangles and without any degree-4 points in the representations are exactly the class of 4-connected planar graphs with triangular internal faces and a non-triangular outerface [12, 17]. However, the same cannot be said about proportional contact representations; there are instances of 4-connected planar graphs with triangular internal faces and a non-triangular outerface that have no proportional contact representations with rectangles for some weight functions. Recently, Eppstein *et al.* [7] characterized the class of *area-universal* rectangular duals, i.e., rectangular duals that can realize any specified area for the rectangles. In summary, the best known lower and upper bounds for hole-free proportional contact representations of 4-connected planar graphs are 4 and 8, respectively [2, 6].

1.2 Our Contributions

We are interested in narrowing the gap between the known lower and upper bounds on the polygonal complexity in proportional contact representations for 4-connected planar graphs in various settings (rectilinear or not, hole-free or with holes). We also present some computational complexity results about a natural recognition problem, and disprove a 10-year old conjecture. To summarize:

(1) We describe an algorithm for constructing a proportional contact representation of a 4-connected internally triangulated planar graph using 6-sided polygons with arbitrarily small cartographic error[1]. We then prove the existence of a representation without cartographic error.

(2) We disprove a conjecture, posed independently by two sets of authors [2, 4], about the existence of a Hamiltonian canonical order (a canonical order that induces a Hamiltonian cycle) in a 4-connected maximal planar graph. In particular, this shows that a method suggested in [2] for constructing 6-sided rectilinear proportional contact representations for 4-connected graphs will not work.

(3) We show that it is **NP**-complete to decide whether a 4-connected planar graph has a proportional contact representation with rectangles.

2 Proportional Contact Representation of Complexity 6

In this section we give an algorithm for constructing 6-sided proportional contact representation of an internally triangulated 4-connected planar graph with arbitrarily small cartographic error. With the help of this algorithm, we then prove the existence of a representation with complexity 6 and no cartographic error.

2.1 Representations with Cartographic Error

We prove the following main theorem in the rest of this section.

Theorem 1. *Let $G = (V, E)$ be an internally triangulated 4-connected plane graph and let $w : V \mapsto \mathbb{R}^+$ be a weight function on the vertices of G. Then for any $\epsilon > 0$, G has a contact representation where each vertex v of G is represented by a 6-sided polygon with area in the range $[w(v), w(v) + \epsilon]$.*

Assume first that the outer face of G is not a triangle. Then G admits a *rectangular dual* Γ (a rectilinear dual that uses only rectangles) due to [12, 17]. We need the following definitions for a rectangular dual Γ of G. The graph G is the *dual* of Γ. Since each polygons in Γ is a rectangle, the adjacency between two rectangles in Γ representing an edge of G can occur in one of two ways: (i) through a shared horizontal segment (*horizontal adjacency*), or (ii) through a shared vertical segment (*vertical adjacency*). We say Γ is *topologically equivalent* to another rectangular layout Γ' when both Γ and Γ' have the same dual graph G and each edge of G is represented by the same type of adjacency (horizontal or vertical) in both Γ and Γ'. A *line-segment* in Γ is the union of inner edges of Γ forming a consecutive part of a straight-line. A line-segment not contained in any other line-segment is *maximal*. A maximal line-segment s is called *one-sided* if it forms a full side of at least one rectangular face, or in other words, if the perpendicular line segments that attach to its interior are all on one side of s. Otherwise, s is *two-sided*. Eppstein *et al.* proved that if all the maximal segments in a rectangular dual Γ are one-sided then Γ is *area-universal*, which means that any distribution of areas to the rectangles in Γ can be realized with a topologically equivalent layout [7]. Unfortunately, not every internally triangulated 4-connected plane graph has a rectangular dual that is also area-universal. With the next lemma we can prove a slightly

[1] *Cartographic error* in a representation of a graph G is the maximum over the vertices v in G of the value $|A(v) - w(v)|$, where $A(v)$ is the area for v and $w(v)$ is its weight.

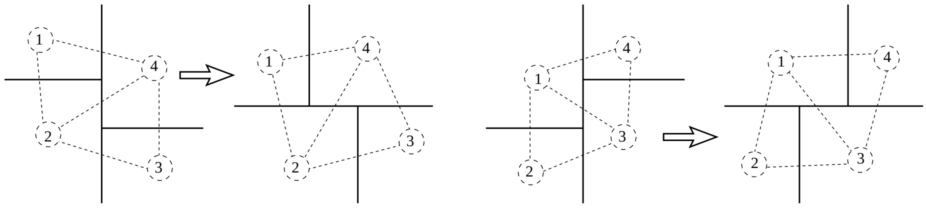

Fig. 1. Illustration for the proof of Lemma 1.

weaker statement which can help us reduce the polygonal complexity. Specifically, we can show that for any such graph with non-triangle outerface, there exists a rectangular dual where all two-sided maximal segments are horizontal.

Lemma 1. *Let G be an internally triangulated 4-connected plane graph with a non-triangle outerface. Then G has a rectangular dual with no vertical two-sided segment.*

Proof Sketch. We start by computing a (possibly two-sided) rectangular dual Γ of G [12]. If G is one-sided or has only horizontal 2-sided maximal segments we are done. Let s be a vertical maximal segment in Γ. Call every degree-3 point on s a *junction* on s. If s is not one-sided, then going from bottom to top along s there will at least one of the two configurations in Fig. 1. In both cases, we modify the layout locally as illustrated in Fig. 1. If we repeatedly apply this operation for each vertical two-sided segment in Γ, there will be no more vertical two-sided segment (details in the Appendix). □

Once we have a rectangular dual of a planar graph G with no two-sided vertical maximal segments, we modify the representation into a contact representation with 6-sided polygons to realize any set of weights on the vertices of G, at the expense of ϵ-cartographic error, $\epsilon > 0$.

Lemma 2. *Let $G = (V, E)$ be an internally triangulated plane graph and let Γ be a rectangular dual of G such that Γ contains no vertical two-sided maximal segment. Then G admits contact representation Λ such that each vertex v of G is represented by a polygon of complexity at most 6 with area in the range $[w(v), w(v) + \epsilon]$, where $w : V \mapsto \mathbb{R}^+$ is an arbitrary weight function and $\epsilon > 0$.*

Proof Sketch. If Γ contains no two-sided maximal segment, then it can realize any weight function [7] and we are done. Otherwise, for each horizontal two-sided maximal segment s, we replace s by a rectangle with a small height (< the minimum feature size of Γ) and with horizontal span same as s; see Fig. 2(a)–(b). It is easy to see that this modification makes all maximal segments one-sided, which makes the resulting representation Γ' area-universal. Let Γ'' be a rectangular layout, where the area of each newly formed rectangle is ϵ and the areas for all other rectangles realize the weight function w; see Fig. 2(c). Suppose s was a horizontal two-sided segment in Γ and let l and l' be the top and bottom side of the corresponding rectangle in Γ' (also in Γ''). We select some points on l' corresponding to all the junctions on l so that the order of all these junctions defined by s is respected. We then add an edge from each junction on l to its corresponding point on l'. In this way the area of the rectangle defined by l and l'

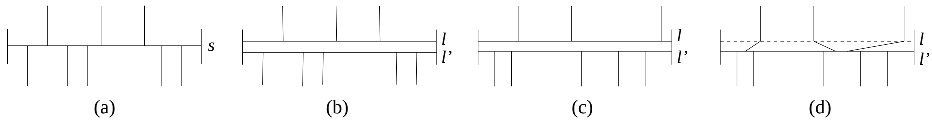

Fig. 2. Illustration for the proof of Lemma 2.

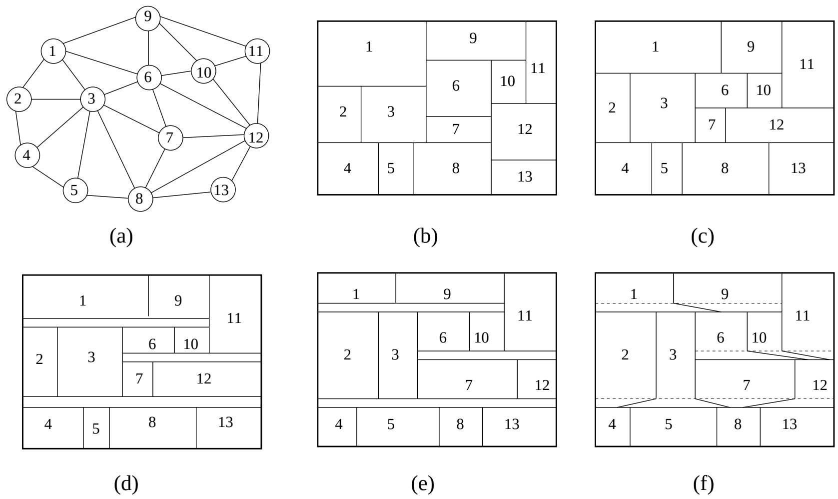

Fig. 3. Illustration of the construction with 6-sided polygons and ϵ-cartographic error.

which had area ϵ is distributed among the rectangles above l. For each rectangle at most two additional corners are thus added making it a polygon with at most 6 sides. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. If the outerface of G is a not triangle, then by Lemma 1 and Lemma 2, G admits a desired representation. We thus assume that the outerface of G is a triangle abc. Then G admits no rectangular dual. However, G admits a rectilinear dual where one of the outer vertices, say a, is represented by a 6-sided "L-shaped" rectilinear polygon and all other vertices are represented by rectangles [15]. Furthermore, the representation is contained inside a rectangle and so is the union of the rectangles representing all vertices of $G' = G - \{a\}$. We then obtain a contact representation of G' with 6-sided polygons by Lemma 1 and Lemma 2. Since the boundary of the rectangular dual of G' is not changed, we can sill add the **L**-shaped polygon for a around it with the desired area and correct adjacencies. □

Figure 3 illustrates the construction of a contact representation of a 4-connected plane graph G with 6-sided polygons using the above procedure.

2.2 Representations without Cartographic Error

Here we prove that an internally triangulated 4-connected plane graph has a proportional contact representation of complexity 6 with no cartographic error for any weight function. We begin with a representation that has ϵ-cartographic error and argue that it can be modified so as to remove the errors, while preserving the topology of the layout.

Let $G = (V, E)$ be a graph with weight function $w : V \mapsto \mathbb{R}^+$. Consider a vertex order $v_1, v_2, \ldots, v_n$ of the vertices of G. Let $\Lambda = \langle \lambda_1, \lambda_2, \ldots, \lambda_n \rangle$ be a list of n non-negative real numbers.Then the Λ-*vicinity* of w is the set of all weight functions $w' : V \mapsto \mathbb{R}^+$ such that $|w(v_i) - w'(v_i)| <= \lambda_i$ for each vertex v_i of G. If $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \lambda$, then the Λ-vicinity of w is also called the λ-*vicinity* of w. We have the following lemma, whose proof is included in the appendix.

Lemma 3. *Let $G = (V, E)$ be an internally triangulated plane graph and let Γ be a rectangular cartogram of G for a weight function $w : V \mapsto \mathbb{R}^+$ where Γ contains no vertical two-sided segment. Then there exists a sufficiently small $\lambda > 0$ such that for any weight function $w' : V \mapsto \mathbb{R}^+$ in the λ-vicinity of w, there is a rectangular cartogram $\Gamma(w')$ of G with respect to w', where $\Gamma(w')$ is topologically equivalent to Γ.*

Using the above lemma, we can prove the existence of a cartogram for a internally triangulated 4-connected graph without any cartographic error.

Theorem 2. *Let $G = (V, E)$ be a internally triangulated 4-connected plane graph and let $w : V \mapsto \mathbb{R}^+$ be a weight function on the vertices of G. Then G has a proportional contact representation of complexity 6 with respect to w.*

Proof. Here we use the algorithm from Theorem 1. We assume that G has a non-triangle outerface since the case with triangle outerface can be handled in the same way as in Theorem 1. Let Γ be a representation of G obtained by this algorithm. Each polyline between two horizontal segments in Γ consists of two segments: a vertical segment followed by a segment with an arbitrary slope. Call each such polyline a *vertical 2-line* and call the common point between the two segments of a vertical 2-line a *pivot point*. For example, in Fig. 2(d), there are three such vertical 2-lines. Each pivot point can be moved up or down, resulting in the increase or decrease of the areas of its adjacent polygons. We use this flexibility to find a representation without cartographic error.

Let Γ_0 denote the rectangular representation when we set $\epsilon = 0$, where each pivot point has the same y-coordinate as the bottommost point of the corresponding vertical 2-line. Note that Γ_0 may no longer represent G because the adjacencies between rectangles on opposite sides of a horizontal segment may change. We now modify the weights for the rectangles in Γ_0 so that the total weights for all rectangles remains the same and we can move the pivot points to correct the error created by this change. We choose the new weight function w' to be in the λ-vicinity of w, for some $\lambda > 0$ by applying Lemma 3, so that we can find rectangular cartogram Γ' topologically equivalent to Γ_0. We set the width and height of this rectangular representation to be B and H, where $B \times H = \sum_{v \in V} w(v)$. Then $B_{min} = (w_{max} + \lambda)/H$ and $H_{min} = (w_{max} + \lambda)/B$ are the minimum width and the minimum height of a rectangle in Γ', respectively.

We now define the weights for all the rectangles above each two-sided horizontal maximal segment s as follows. Let us consider set S of all the vertical 2-lines incident to the top-side of a particular rectangle R. Fig. 4 illustrates a set of such vertical 2-lines where for convenience of visualization the pivot points and the bottommost points are placed at different y-coordinates. Let x_l and x_r be the x-coordinates of the left and right

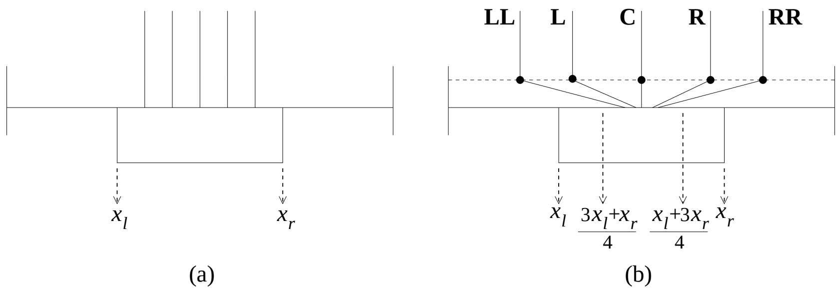

Fig. 4. Illustration for Theorem 2.

side of the rectangle to the bottom of the horizontal segment. In the final representation with 6-sided polygons, we want the bottommost point of each of these vertical 2-lines to be placed between the one-fourth and the three-fourth of its horizontal span; i.e., between $(3x_l + x_r)/4$ and $(x_l + 3x_r)/4$. In this respect we partition the vertical 2-lines in S into five classes as follows, depending on the x-coordinates of the pivot points; see Fig. 4(b) (pivot points are highlighted by black dots).

(i) **LL**-*lines*: pivot points are to the left of x_l
(ii) **L**-*lines*: pivot points are between x_l and $(3x_l + x_r)/4$
(iii) **C**-*lines*: pivot points are between $(3x_l + x_r)/4$ and $(x_l + 3x_r)/4$
(iv) **R**-*lines*: pivot points are between $(x_l + 3x_r)/4$ and x_r
(v) **RR**-*lines*: pivot points are to the right of x_r

By a left-to-right scan, we set the weight for all the rectangles to the left of a **LL**-line and a **L**-line in S. For each **LL**-line, we set the weight of the rectangle to its left such that the total weight for all the rectangles to its left is decreased by an amount $\epsilon > 0$ where $\epsilon < \lambda$ and $\epsilon < (B_{min} \times H_{min})/8$. For each **L**-line, we set the weight of the rectangle to its left so that the weight of all rectangles to its left is decreased by $\epsilon > 0$ where $\epsilon < \lambda$ and $\epsilon < (B_{min} \times H_{min})/(8|S|)$. Similarly by a right-to-left scan, we set the weight for each the rectangle to the left of a **RR**-line (resp. **R**-line) so that the weight of all rectangles to the left of the polyline is increased by $\epsilon > 0$ where $\epsilon < \lambda$ and $\epsilon < (B_{min} \times H_{min})/8$ (resp. $\epsilon < (B_{min} \times H_{min})/(8|S|)$). For each **C**-line, we set the weight of the rectangle to its left so that the total weight of all the rectangles to its left remains the same. Once we compute ϵ for all the vertical 2-lines, we can then compute the exact weights to be assigned to each rectangle in S. By Lemma 3, we then find a rectangular cartogram Γ' with the new weight function such that Γ' is topologically equivalent to Γ. By shifting the pivot points up as needed, we can realize exact weights for each rectangle. Note that by the weight distribution, it is never required to shift any pivot point more than H_{min} distance, which is the minimum vertical distance between two horizontal segments. By choosing ϵ small enough, we can make sure that the segments between the pivot points and bottommost points of two vertical 2-lines do not cross. □

3 Hamiltonian Canonical Cycles

Let $G = (V, E)$ be a maximal plane graph with outer vertices u, v, w in clockwise order. A *canonical order* of the vertices $v_1 = u$, $v_2 = v$, v_3, . . ., $v_n = w$ of G, is one that meets the following criteria for every $4 \leq i \leq n$, where the G_i is the subgraph of G induced by the vertices v_1, v_2, . . ., v_{i-1}:

- $G_{i-1} \subseteq G$ is biconnected, and the boundary of its outerface is a cycle C_{i-1} containing the edge (u, v).
- vertex v_i is in the outerface of G_{i-1}, and its neighbors in G_{i-1} form an (at least 2-element) subinterval of the path $C_{i-1} - (u, v)$.

A *Hamiltonian Canonical Cycle* in a maximal planar graph G is a canonical order $v_1, v_2, \ldots, v_n$ of the vertices of G such that $v_1 v_2 \ldots v_n$ is also a Hamiltonian cycle of G. Whether every 4-connected maximal planar graph has a Hamiltonian canonical cycle is a question asked at least two times by two sets of authors in different contexts [2, 4]. In fact, an algorithm in [2] produces 6-sided rectilinear proportional contact representation of any maximal planar graph that has a Hamiltonian canonical cycle. If the above conjecture were true, that would suffice to lower the current best known upper bound on the polygonal complexity for 4-connected maximal planar graphs from 8 sides to 6 sides. Unfortunately, we show that the conjecture is not true by constructing a 4-connected maximal planar graph with no Hamiltonian canonical cycle.

Theorem 3. *There exist 4-connected maximal planar graphs that do not have any Hamiltonian canonical cycle in any embedding.*

To prove this claim we construct a 4-connected graph G where there are two internal faces of length 4 and the remaining faces (including the outerface) are triangles. We put one isomorphic copy of the graph K in Fig. 5 inside each of the faces of length 4 in G (so that the four vertices on the faces are superimposed with a, b, c and d). In any embedding of this graph, at least one copy of K will retain its embedding. Thus in order to prove Theorem 3, it suffices to prove that there is no Hamiltonian canonical cycle of of any plane graph that contains an isomorphic copy of K with the fixed embedding.

Lemma 4. *Let G be a maximal plane graph with an isomorphic copy of the graph K in Fig. 5 as an embedded subgraph. Then G has no Hamiltonian canonical cycle.*

The graph K is very symmetric and it contains four isomorphic copies of the graph L in Fig. 6. Lemma 5 shows that any Hamiltonian canonical cycle of K must follow a restricted path inside L. This can be used to prove Lemma 4.

Let $\mathcal{C}$ =v_1, v_2, . . ., v_n be a Hamiltonian canonical cycle of a maximal planar graph G. Since, $\mathcal{C}$ induces a canonical order, we can consider it as a directed cycle where the edge (v_i, v_{i+1}) is directed towards v_{i+1} for $1 \leq i \leq n - 1$ and the edge (v_n, v_1) is directed towards v_1. This also induces a direction for any subpath of $\mathcal{C}$.

Lemma 5. *Let G be a maximal plane graph that contains an isomorphic copy of graph L of Fig. 6 as an embedded subgraph. Then there exists no Hamiltonian canonical cycle $\mathcal{C}$ of G such that it contains a subpath P where:*

(i) the first two vertices on P are from the set $\{A, B\}$,

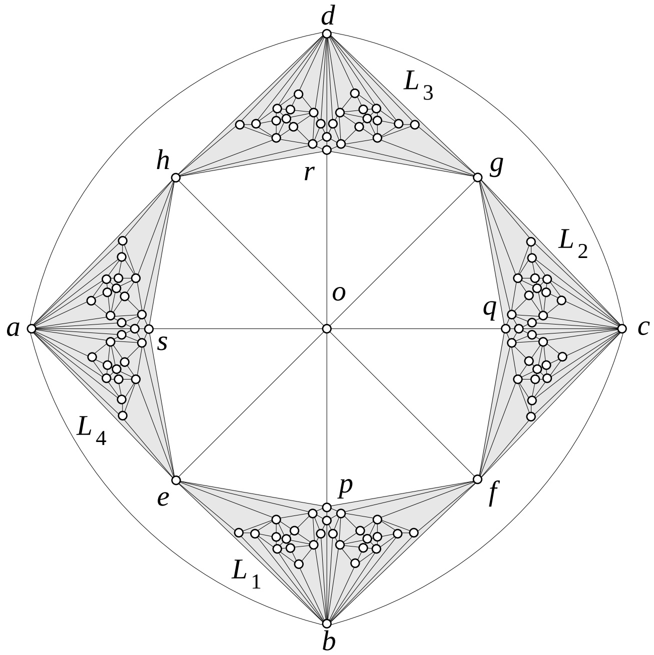

Fig. 5. The graph K used in Theorem 3 and Lemma 4.

(ii) the third vertex on P is also from the subgraph L, and
(iii) either the last vertex of P is Z and F is after Z in $\mathcal{C}$ or the last vertex of P is F.

Proof Sketch. Assume for a contradiction that there exists a Hamiltonian canonical cycle $\mathcal{C}$ of G with a subpath P such that the conditions (i)–(iii) hold. Denote the vertex set of L by S. First note that the vertex set $T = \{A, B, Z, F\}$ defines a separating cycle of G. Let L' be the graph induced by $S-T$. Then P can enter and exit L' only once. The first two vertices on P are A and B. Without loss of generality, we may assume that a_1 is the third vertex. Since C is the only vertex for P to go between vertices on different sides of the polyline BCF, C must appear on P after all the vertices from the left of BCF have appeared on P. Then since $\mathcal{C}$ is a Hamiltonian cycle as well as a canonical order, a careful observation shows that the initial subpath of P is the one drawn by the thick black polyline, followed by the one drawn by the dotted grey polyline in Fig. 6. On the other hand, the subpath of P ending at C must be the one drawn by the thick gray solid polyline; see Fig. 6. However, now there is no edge to go between these two subpaths, which is a contradiction. A detailed proof is in the appendix. □

We are now ready to prove Lemma 4. We are giving the proof sketch here. The details are in the appendix.

Proof Sketch of Lemma 4. Assume for a contradiction that G has a Hamiltonian canonical cycle $\mathcal{C}$. For any embedded subgraph H of G bounded by a separating cycle C_H, call the subgraph induced by the vertices of $H - C_H$ the *inside* of H. Since $\mathcal{C}$ is a canonical order, withour loss of generality the first vertex on $\mathcal{C}$ inside K is e. Then both

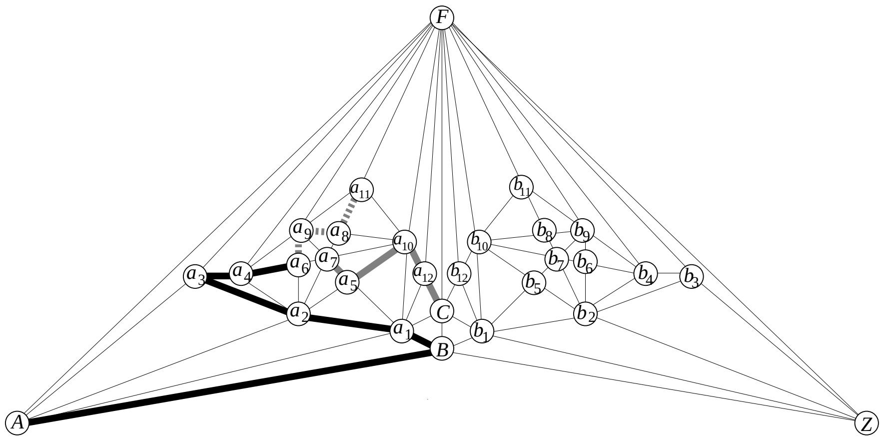

Fig. 6. The graph L used in Lemmas 4 and 5.

a and b appears before e and $\mathcal{C}$ can enter and exit the inside of K only once and either c of d is the only exit vertex. Call the subpath of $\mathcal{C}$ between the entry and the exit vertex P. Assume due to symmetry that P enters in the inside of L_1 after e. Since either c or d is the exit vertex, P must enter and exit the inside of L_4 through s and h, respectively and o must immediately preceed s on P. Looking back at L_1, in case P goes from either p or f to o, it must go from h to the inside of L_3 and eventually exit the inside of K through c. However, this is not possible by Lemma 5. Thus P must visit all the vertices of L_1 and exit through f and go via the inside of L_2, to c. Since a, b, c have all been already visited by P, it must then exit though d from the inside of K;, which is also not possible due to Lemma 5. □

The proof of Theorem 3 follows from Lemma 4 since we could construct a graph that contains an isomorphoc copy the graph K as an embedded graph.

4 NP-Hardness for Rectangular Representations

In this section, we consider the following problem. Given a 4-connected plane graph $G = (V, E)$ with triangle and quadrangle internal faces and a non-triangle outerface and a weight function $w : V \mapsto \mathbb{R}^+$ on the vertices of G, we want to determine whether G has a rectangular cartogram with respect to w. Let us call the problem **RectangleCartogram** (**RC**). Clearly this problem is in **NP** since given a rectangular dual Γ of G, one can verify in linear time whether each rectangle in Γ has the desired area and whether the topology of Γ matches that of G. We now show that this problem is **NP**-complete by a reduction from the well-known **NP**-hard problem **Partition** defined as follows. Given a set of integers $S = \{x_1, \ldots, x_n\}$ with $\sum_{i=1}^{n} x_n = 2A$ for some integer A, we want to find a subset I of S such that $\sum_{x_i \in I} x_i = A$.

Given an instance of **Partition**, we construct an instance of **RC** as follows. For each integer x_i of S, we have a subgraph consisting of eight vertices X_i, p_i, p_{i+1}, q_i, q_{i+1}, a_i, b_i and c_i. We highlight such a subgraph for x_i in Fig. 7. Note that the constructed

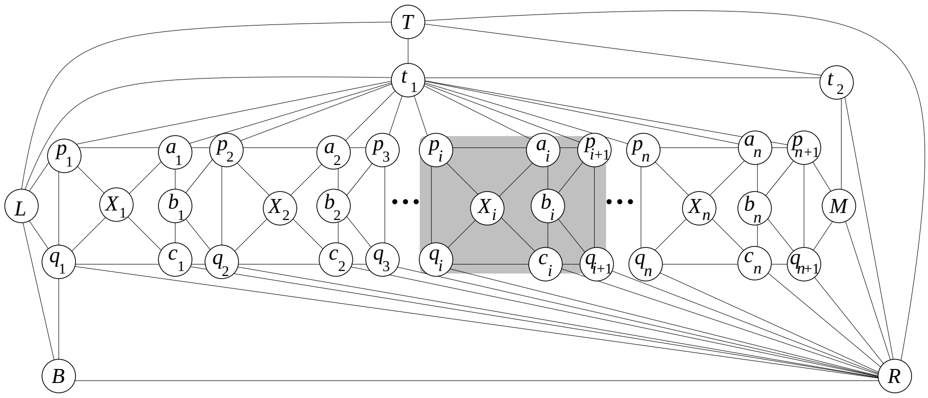

Fig. 7. The graph constructed from an instance of a **Partition** problem.

graph is 4-connected with a nontriangle outerface where each internal faces is either a triangle or a quadrangle.

We define the weight function as follows. Define $m = \min_{i=1}^{n} x_n$. For each vertex X_i, we define $w(X_i) = x_i$. We give a very small weight, say $\delta = m/20$ to each vertex a_i, b_i, c_i, p_i, q_i for $1 \leq i \leq n$ and to each vertex L, R, T and B. We give a very large weight $W(M)$ to the vertex M such that $\sqrt{W(M) + 2A + (n+1)\delta} - \sqrt{W(M)} < \sqrt{m}$. Finally we give weights to the vertices t_1 and t_2 such that $w(t_1) : w(t_2) = \sqrt{W(M) + 2A + (n+1)\delta} - \sqrt{W(M)} : \sqrt{W(M)}$.

We now have the following lemma, whose proof is included in the appendix.

Lemma 6. *There exists a subset I of S such that $\sum_{x_i \in I} x_i = A$ if and only if there is a cartogram of G with respect to the weight function w.*

We can thus reduce an instance S of Problem **Partition** to an instance (G, w) of Problem **RectangleCartogram** such that S has a solution if and only if G has a rectilinear cartogram with respect to w. This yields following theorem.

Theorem 4. *Problem **RectangleCartogram** is* **NP***-complete.*

5 Conclusions and Open Problems

We addressed the problem of proportional contact representation of 4-connected internally triangulated planar graphs and showed that non-rectilinear polygons with complexity 6 are sufficient. Narrowing the gap between this upper bound and the currently best known lower bound of 4 remains open. With the additional restriction of using rectilinear polygons, there are instances where complexity 6 is required and here we showed that it is **NP**-complete to decide whether a 4-connected planar graph with triangle and quadrangle internal faces admits a representation with rectangles. It will be interesting to investigate the complexity of the same problem for internally triangulated graphs. On the other hand currently best known upper and lower bounds for rectilinear cartograms of 4-connected internally triangulated planar graphs are 8 and 6, respectively. Thus to find out whether the true bound in this case is 6 or 8 is open.

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Appendix

Proof of Lemma 1. Let Γ be a rectangular dual of G. The fact that G admits one follows from several (independent) results from the literature [12, 17]. We will now modify Γ in such a way that it no longer contains a vertical two-sided maximal segment. Let s be a maximal segment in Γ. Since G is internally triangulated plane graph, there is no degree-4 point on s. Call every degree-3 point on s a *junction* on s. If s is a vertical segment, then a junction p on s is either *left-side* or *right-sided* depending on whether the perpendicular segment attached to s at p comes from the left side or the right side of s. Suppose s is a vertical maximal segment. If the junctions on s are all left-sided or all right-sided, then s in one-sided. Otherwise, going from bottom to top along s there will be (i) a left-sided junction followed by a right-sided junction; or (ii) a right-sided junction followed by a left sided junction. Whenever we encounter one of these two cases, we modify s, by splitting it into two smaller vertical segments joined end-to-end by a horizontal segment, as illustrated in Fig, 1. Note that the adjacencies are not disturbed by this modification. If we repeatedly apply this operation for each vertical two-sided maximal segment in Γ, there will be no more vertical two-sided segment. □

Proof of Lemma 2.

If Γ contains no two-sided maximal segment, then by [7], Γ is area-universal and hence it can realize any weight function w by a topologically equivalent layout, which gives the desired representation. Otherwise, we modify Γ to obtain a desired representation. For each horizontal two-sided maximal segment s, we replace s by a rectangle with a small height ($<$ the minimum feature size of Γ) and with horizontal span same as s; see Fig. 2(a)–(b). Suppose the top and the bottom sides of the newly formed rectangle are l and l'. See that for s all the perpendicular segment coming from top is now attached to l and all the perpendicular segment coming from bottom is now attached to l'. Thus both l and l' are one-sided. Also note that this modification does not make any one-sided segment two sided, since the two new segments l and l' are attached to the two vertical segments from the same side as s was attached. Thus, if we do this for each horizontal two-sided maximal segment, all the maximal segments will be one-sided and hence the resulting representation Γ' will be area-universal due to [7]. Let Γ'' be a rectangular layout, topologically equivalent to Γ', where the area of each newly formed rectangle is ϵ and the areas for all other rectangles realizes the weight function w; see Fig. 2(c). From Γ'', we obtain a desired representation as follows.

Suppose s was a horizontal two-sided segment in Γ and let l and l' be the top and bottom side of the corresponding rectangle in Γ' (also in Γ''). Then then junctions in s are split into two sets, one on l and the other on l'. A left-to-right ordering of these junctions are defined by s. Note that this ordering is respected among the two sets on l and l' in Γ''. We now select some points on l' corresponding to all the junctions on l so that the order of all these junctions defined by s is respected. We then add an edge from each junction on l to its corresponding point on l'. In this way the ϵ area of the rectangle for s is divided in parts into the rectangles on top of l. This operation retains the original adjacencies in Γ into Γ''. Note that a rectangle can participate in this operation only for one horizontal segment, which contains its bottom side. For each rectangle at most two additional corners are thus added making it a polygon with at most 6 sides. Furthermore, since a rectangle now gets an additional fraction of ϵ area along with its own area, the area for each polygon representing a vertex v falls in the range $[w(v), w(v) + \epsilon]$. □

Proof of Lemma 3. Assume an arbitrary order $v_1, v_2, \ldots, v_n$ of the vertices of G. Consider any value $\lambda > 0$ and take a fixed weight function w' in the λ-vicinity of w. Let Γ' be the representation of G obtained by the algorithm in Theorem 1, where we set $\epsilon = 0$. Note that in this case each dummy rectangle corresponding to a two-sided horizontal segment will have zero area and hence become a horizontal segment again. Thus if we consider each of these zero-area rectangles only as a horizontal segment, we get a rectangular layout Γ^*, where for each vertex v of G, the rectangle corresponding to v has area $w'(v)$ and all the adjacency in Γ is preserved except that possibly the adjacencies between rectangles on opposite sides of a two-sided horizontal maximal segment may be disturbed (old adjacencies removed, new adjacencies created). Note that the existence of such a layout and its uniqueness for w' was also proved in [18].

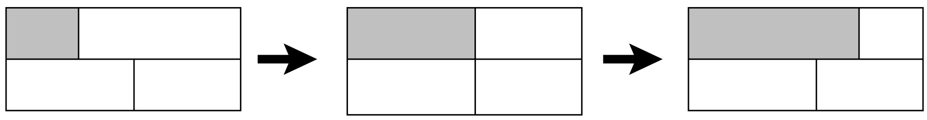

Fig. 8. Illustration for Lemma 3.

We will now show that there exists some sufficiently small $\lambda > 0$ for which, Γ^* is topologically equivalent to Γ. Take a a list of non-negative real numbers $L = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that for any weight function w' in the Λ-vicinity of w, Γ^* remains topologically equivalent to Γ. Note that there exists at least one such list where $\lambda_i = 0$ for $1 \leq i \leq n$. We take such a list Λ^* with the least number of zero values. We now claim that there is no element with zero value in L^*. Assume for a contradiction that there is at least one element $\lambda_i = 0$ in L^*. This implies that for any Γ^* realizing a weight function w' in the L^*-vicinity of w, the area of the rectangle R_i representing a vertex v_i either cannot be increased or cannot be decreased while preserving the topology and the areas of all other rectangles in Γ^*. This is only true if some corners of R_i is a degree-4 points in Γ^*. This is illustrated by a simple example in Fig. 8. Suppose that we want to increase the area of a grey rectangle. Since all two sided segments are horizontal, the topology of the layout can be changed only when two vertical segment touching the same horizontal segment, one from the top and the other from the bottom, change their order of x-coordinates. But as shown in Fig. 8, before the two segment changes order, there is an intermediate step where their x-coordinates are the same, i.e. a degree-4 point is created in the layout. Hence unless there is a degree-4 point, one can always increase or decrease the area of a particular rectangle. However since Γ^* is topologically equivalent to Γ, it has no degree-4 point. Thus the area of R_i can be expanded or shrunk by a small positive amount δ while keeping the areas of the other rectangles fixed and preserving the topology of Γ^*. Let δ_{min} be the smallest over all such weight functions w' in the L^*-vicinity of w. Take a list $L' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ where $\lambda'_i = \delta_{min}$ and $\delta'_j = \delta_j$ for $j \neq i$. From the above discussion, for any weight function in the L'-vicinity of w, Γ^* is topologically equivalent to Γ while L' has one less zero value than L^*. This is a contradiction and hence L^* has no element with a zero value. Choose $\lambda = min_{i=1}^{n} \lambda_i$. Then $\lambda > 0$ and there exists a rectilinear cartogram topologically equivalent to Γ for any weight function in the λ-vicinity of w. □

Proof of Lemma 4. Assume for a contradiction that G has a Hamiltonian canonical cycle $\mathcal{C}$. For any embedded subgraph H of G bounded by a separating cycle C_H, we call the subgraph induced by the vertices of $H - C_H$ the *inside* of H. Since $\mathcal{C}$ is a canonical order, the first vertex on $\mathcal{C}$ inside K must be one of the four vertices e, f, g, h, say e. In that case both a and b must have appeared before e on $\mathcal{C}$. Then $\mathcal{C}$ can enter and exit the inside of K only once and either c of d is the only exit vertex. Call the subpath of $\mathcal{C}$ between the entry and the exit vertex P. Assume due to symmetry that P enters in the inside of L_1 after e. Then P must enter and exit the inside of L_4 through s and h. However it cannot exit though s since in that case it needs to go from the inside of L_1 to the vertex f, then via the inside of L_2 it must go to g and finally via the inside of L_3 it must go to h. This is not possible since the last vertex on P must be either c or d. Thus P must enter and exit the inside of L_4 through s and h, respectively; and o must immediately precede s on P.

Again P must visit all the vertices of L_1 except possibly for f before it can go to any other vertex in the inside of K. In case P goes from either p or f to o, it must go from h via the inside of L_3 to g, then from g via the inside of L_2, it will exit the inside of K through c. However, this is not possible by Lemma 5; a contradiction. Thus P must visit all the vertices of L_1 and exit through f to a vertex of L_2. Next P goes to c from f and it must visit all the vertices of L_2 (except possibly for g) before it finally exits from the inside of L_2. Since a, b, c have all been already visited by P, it must exit though d from the inside of K. Therefore, P goes from e via the inside of L_1 to f, then via the inside of L_2 to o followed by s and then via the inside of L_4 to h. Finally, P must go from h to r and then via the inside of L_4 to d, which is not possible due to Lemma 5. Thus there cannot be any Hamiltonian canonical cycle in G. □

Proof of Lemma 5. Assume for a contradiction that there exists a Hamiltonian canonical cycle $\mathcal{C}$ of G with a subpath P such that the conditions (i)–(iii) hold. Since P is a subpath of a Hamiltonian cycle, a vertex can appear at most once on P. Denote the vertex set of L by S. First note that the vertex set $T = \{A, B, Z, F\}$ defines a separating cycle of G. Let L' be the graph induced by $S - T$. Then P can enter and exit L' only once. The first two vertices on P are A and B. Without loss of generality, we may assume that a_1 is the third vertex. The fourth vertex can be either a_2 or C. Since C is the only vertex for P to go between vertices on different sides of the polyline BCF, the fourth vertex must be a_2. The next two vertices are a_3 and a_4, respectively since a_3 has degree 4 but neither A nor F appears immediately before or after a_3. Since $\mathcal{C}$ is a canonical order, the next vertex must be a_6. This part of P is the thick black polyline in Fig. 6.

We already showed that C must appear on P after all the vertices of L to the left of the polyline BCF have appeared. By a similar logic a_{12} and a_{10} must be the two vertices immediately preceding C. Again a_5 is a vertex of degree 4 and two of its neighbors a_1, a_2 are neither immediately before or after a_5. Thus the two vertices immediately before a_{10} are a_5 and a_7, respectively. This part of P is highlighted by the thick gray solid polyline; see Fig. 6.

Now let us find out the vertices on P between the thick black and the thick gray polylines. Since $\mathcal{C}$ is Hamiltonian, a_7 cannot immediately follow a_6 nor can it appear before all of a_8, a_9 and a_{11} have appeared. Then the the successive vertices on P following a_6 must be a_9, a_8 and a_{11} in this order, since $\mathcal{C}$ is a canonical order. However, now there is no edge to go from a_{11} to the start of the thick gray path; a contradiction. □

Proof of Lemma 6. Suppose first that there is a cartogram Γ of G with respect to w. In any such cartogram, the four vertices L, T, R and B defines a rectangular frame F' inside which the rectangles for the remaining vertices lie. Without loss of generality, we assume that the rectangles L, T, R and B are to the left, top, right and bottom of F', respectively, illustrated in Fig 9. Since the face t_1, t_2, M and p_n forms a rectangle, this face will correspond to a point of degree four in Γ incident to rectangles for the four vertices. Thus the rectangles $R(t_1)$ and $R(t_2)$ for the vertices t_1 and t_2, respectively, have the same height and together they span the top of the frame F'. Define the rectangular frame $F = F' - R(t_1) - R(t_2)$. Scale Γ such that F is a square. Then from the weight distribution of t_1 and t_2, the rectangle $R(M)$ for M is also a square. The left and bottom of this square define two lines l_v and l_h, respectively, illustrated in Fig. 9. Now the rectangle for each vertex X_i lies either to the left of l_v or to the bottom of l_h, but not both since $\sqrt{W(M) + 2A + (n+1)\delta} - \sqrt{W(M)} < \sqrt{m} \le \sqrt{x_i}$. Thus each subgraph for x_i has a drawing in one of two configurations, illustrated in Fig. 9(a) and (b), respectively. Then the vertices corresponding to the rectangles X_i lying to the left of l_v forms the subset I of S. The reason is as follows. Define the four rectangles R_1, R_2, R_3, R_4 that partitions F in this way: $R_1 = R(M)$, R_2, R_3 are the rectangles to the left of l_v; R_2 to the top and R_3 to the bottom of l_h and R_4 is the rectangle to the right of l_v and to the bottom of l_h. Then the area of $R_2 + R_3$ and $R_4 + R_3$ are equal and neither can contain rectangles with weights $A + m$ since the area of R_4 is less than m. Thus the rectangles X_i lying in $R_2 + R_3$ has area A.

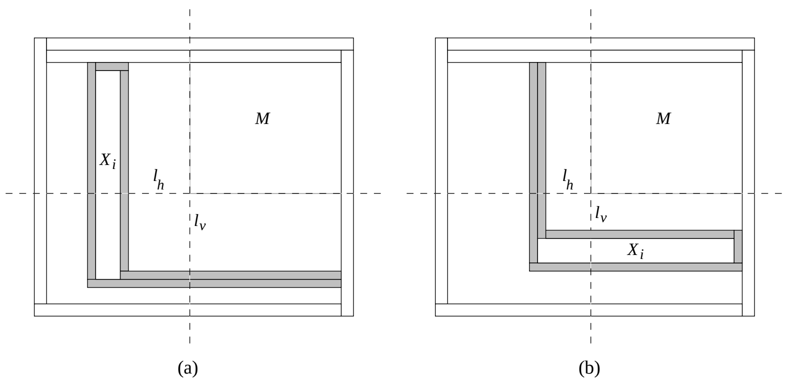

Fig. 9. Illustration of the proof of Lemma 6.

Conversely if we are given a subset I of S such that $\sum_{x_i \in I} x_i = A$, then we construct a cartogram of G as follows. We draw the rectangles for L, T, R, B, t_1 and t_2 such that they enclose a square frame F of size $W(M) + 2A + (n+1)\delta$, and the rectangles L, T, R and B are to the left, top, right, and bottom of the drawing, respectively. We also draw the square $R(M)$ for M to the right-top corner of F. From the weight distribution, the top-left corner of $R(M)$ will be a point of degree four. We then draw the subgraphs for x_i in the way illustrated in Fig. 9(a) if $x_i \in I$; otherwise we draw this subgraph in the way illustrated in Fig. 9(b). Thus we obtain a cartogram of G, as in Fig. 10. Here $x_1, x_2 \in I$, $x_3 \notin I$ and so on. □

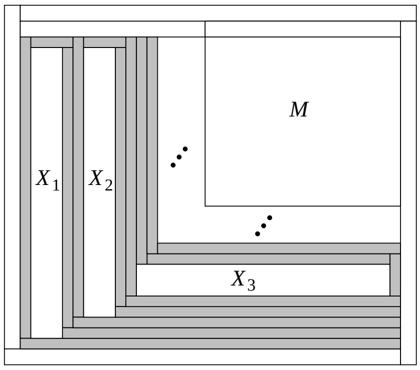

Fig. 10. Construction of a cartogram from a solution of Problem **Partition**.