On Representing Graphs by Touching Cuboids

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Abstract. We consider contact representations of graphs where vertices are represented by cuboids, i.e. interior-disjoint axis-aligned boxes in 3D space. Edges are represented by a proper contact between the cuboids representing their end-vertices. Two cuboids make a proper contact if they intersect and their intersection is a non-zero area rectangle contained in the boundary of both. We study representations where all cuboids are unit cubes, where they are cubes of different sizes, and where they are axis-aligned 3D boxes. We prove that it is NP-complete to decide whether a graph admits a proper contact representation by unit cubes. We also describe algorithms that compute proper contact representations of varying size cubes for relevant graph families. Finally, we give two new simple proofs of a theorem by Thomassen stating that all planar graphs have a proper contact representation by touching cuboids.

1 Introduction

There is a large body of work about representing planar graphs as contact graphs, i.e., graphs whose vertices are represented by geometrical objects with edges corresponding to two objects touching in some specified fashion [1, 10]. Typical classes of objects might be curves, line segments, or polygons. An early result is Koebe's 1936 theorem [11] stating that all planar graphs can be represented by touching disks.

In this paper we consider contact representations of graphs with vertices represented by interior-disjoint axis-aligned boxes in 3D space, called *cuboids*. The graph edges are represented by contacts between the corresponding cuboids, i.e. there is an edge if and only if two cuboids *touch*. We say that two cuboids touch if they intersect and their intersection is contained in the boundary of both. In particular, we are interested in the class of planar graphs represented by *proper contact of cuboids*, where contacts must always have non-zero area. *Cubes* are special cuboids where all sides have the same length.

1.1 Related Work

Cuboids become rectangles in the 2D version of this problem, and such a representation is known as a *rectangular dual* of the input planar graph. There are several (in-

dependent) characterizations of the class of planar graphs that allows such *rectangular duals* [12, 13, 17]. A historical overview and a summary of the state of the art in the rectangle contact graphs literature can be found in Buchsbaum et al. [3] and Felsner [8].

In the 3D case Thomassen [16] shows that any planar graph has a proper contact representation by cuboids. Felsner and Francis [9] prove that any planar graph has a non-proper contact representation by cubes, i.e. a representation where a valid contact between cubes that represent adjacent vertices may have zero area. This naturally raises the question about whether a proper contact representation by cubes is possible for all planar graphs.

1.2   Our Contributions

We study proper contact representations by cuboids, considering cuboids of the same size and of different sizes. We address both combinatorial and computational questions summarized as follows:

- We show that any graph with n vertices that admits a proper contact representation by unit cubes has at most $7n - \Omega(n^{\frac{2}{3}})$ edges, and we also show that such a bound is tight. We describe different families of planar graphs with and without proper contact representation by unit cubes (Section 2).
- We investigate the complexity of the problem of deciding whether a graph admits a proper contact representation by unit cubes. We prove that this problem is NP-complete (Section 3).
- We show that some classes of graphs have no proper contact representation even if cubes are of varying sizes. We also show that varying size cubes allow us to represent larger classes, such as partial planar 3-trees, than unit cubes (Section 4). To the best of our knowledge this is the first significant class of planar graphs for which a proper contact representation by cubes is shown to always exist.
- Finally, we consider general cuboids and describe two new proofs of Thomassen's result. The first one uses the canonical ordering of de Fraysseix, Pach and Pollack [6]; the second one uses Schnyder's realizers [15] (Section 5).

Due to space limitations some proofs are moved to the appendix.

2   Properties of Unit Cube Representations

In this section we establish basic combinatorial properties of graphs with proper contact representations by unit cubes. We start by studying the classical Turán-type problem of finding an upper bound on the number of edges as a function of the number of vertices. We then give examples of graphs that have and don't admit a proper contact representation by unit cubes.

We remark that two axis-aligned unit cubes are said to be in *proper contact* if their interiors have empty intersection and their boundaries intersect in a rectangle of positive area. Two cubes make a *vertex contact* if each cube has a vertex v lying in the interior of a face of the other cube; in such a case v *participates* in a vertex contact. Two cubes make an *edge contact* if each cube has an edge e lying in the interior of a face of the other cube; in such a case the endpoints of e *participate* in an edge contact. Two cubes make a *face contact* if each cube has a face f coincident with a face of the other cube; in such a case the vertices of f *participate* in a face contact.

Theorem 1. The maximum vertex degree of a graph that admits a proper contact representation by unit cubes is 14. Also, in any proper contact representation by unit cubes Γ, if a cube C of Γ has degree 14 then it touches its neighboring unit cubes as follows:

- Two opposite faces of C have 4 vertex contacts each.
- Of the remaining 4 faces, 2 opposite ones have 2 edge contacts each and the other 2 have a face contact each.

Proof: We label each vertex v of C with the multi-set of contact types it participates in: $\mathfrak{v}$, $\mathfrak{e}$, and $\mathfrak{f}$ for vertex, edge, and face contact, respectively. If v participates in a vertex contact, it cannot participate in another and it can participate in at most one edge contact and one face contact, or in two face contacts. If v does not participate in a vertex contact and it does participate in an edge contact, it can participate in at most one more edge contact and one face contact, or in two face contacts. If v only participates in face contacts, it can participate in at most three of them. Thus a valid label of v is a subset of $\{\mathfrak{v}, \mathfrak{e}, \mathfrak{f}\}, \{\mathfrak{v}, \mathfrak{f}, \mathfrak{f}\}, \{\mathfrak{e}, \mathfrak{e}, \mathfrak{f}\}, \{\mathfrak{e}, \mathfrak{f}, \mathfrak{f}\}$, or $\{\mathfrak{f}, \mathfrak{f}, \mathfrak{f}\}$. Every $\mathfrak{e}$ label at vertex v implies another $\mathfrak{e}$ label at an adjacent vertex caused by the same cube touching C. Thus, the total number of $\mathfrak{e}$ labels at vertices of C is twice the number of adjacent cubes contributing to those labels, i.e., each $\mathfrak{e}$ label contributes $\frac{1}{2}$ to the number of cubes adjacent to C. Every $\mathfrak{f}$ label implies three other $\mathfrak{f}$ labels caused by the same cube touching C, so each $\mathfrak{f}$ label contributes $\frac{1}{4}$ to the degree of C. Every $\mathfrak{v}$ label contributes 1 to the degree of C. The total contribution to the degree of C given by a valid vertex label is at most 1.75, by $\{\mathfrak{v}, \mathfrak{e}, \mathfrak{f}\}$. Since C has 8 vertices, it can have degree at most 14.

The only way to obtain degree 14 is if every vertex has label $\{\mathfrak{v}, \mathfrak{e}, \mathfrak{f}\}$. That means two opposite faces of C have face contact. None of the edges connecting these two faces can be part of an edge contact, as otherwise the endpoints of the edge could not be part of a vertex contact. Since the edges that are part of an edge contact form a matching (i.e., no endpoint of a cube participates in two edge contacts), there exist two parallel planes $\mathcal{P}_1$ and $\mathcal{P}_2$, each one containing a face of each cube that makes an edge contact or a face contact. The remaining 8 cubes that form the vertex contacts thus contact C in two opposite faces: 4 touching one face (lying in $\mathcal{P}_1$) and 4 touching the opposite face (lying in $\mathcal{P}_2$). Fig. 1 shows a realization of a degree 14 cube. ■
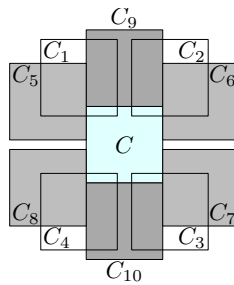

Fig. 1. An example of a degree 14 cube C viewed from above. We show only the outline of the four adjacent cubes C_1, C_2, C_3, and C_4 touching the top face of C. The corresponding cubes C_{11}, C_{12}, C_{13}, and C_{14} touching the bottom face of C are not shown.
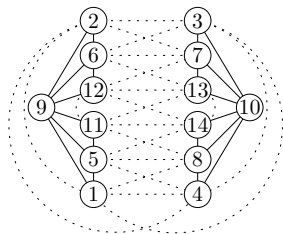
Fig. 2. The graph $\hat{G}_v$ in the statement of Corollary 2. The vertex labels are consistent with the subscripts in Figure 1.
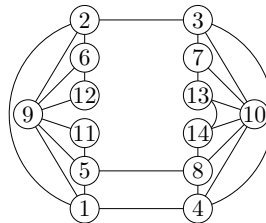
Fig. 3. The subgraph of $\hat{G}_v$ used to construct the seed graph.

A consequence of the proof of Theorem 1 is that any proper contact representation by unit cubes that has a cube C of degree 14, the 8 vertices of C that participate in vertex contacts on opposite faces of C lie on two parallel planes $\mathcal{P}_1$ and $\mathcal{P}_2$ such that C and all its remaining neighbors lie between these two planes. We therefore have three layers: The *middle layer* lies between $\mathcal{P}_1$ and $\mathcal{P}_2$; the *extremal layers* contain the eight cubes making vertex contact with C. In general, we say that a subdrawing of a proper contact representation by unit cubes is *extremal* if it consists of cubes that all have a face lying on a same plane, which separates such cubes from the rest of the representation. A subgraph G' of a graph G is *extremal* if in every proper contact representation by unit cubes of G, the subdrawing of G' is extremal.

We have the following corollary of Theorem 1.

Corollary 1. In a proper contact representation by unit cubes that has a cube C of degree fourteen, the 8 cubes that have vertex contacts on opposite faces of C induce two extremal subdrawings.

By looking at the contacts between the cubes representing the vertices adjacent to a vertex v of degree 14, we obtain the following.

Corollary 2. Let C be a cube representing a degree fourteen vertex v in a proper contact representation by unit cubes of G and let G_v be the graph induced by the neighbors of v (not including v). G_v is a subgraph of the graph $\hat{G}_v$ of Fig. 2. More precisely, G_v has the same vertex set as $\hat{G}_v$, it includes all solid edges, and a (possibly empty) subset of the dotted edges of Fig. 2. Also, if G_v includes a dotted edge e, it does not include any other dotted edge crossing e in Fig. 2.

We are now ready to establish an upper bound to the number of edges in a graph that admits a proper contact representation by unit cubes.

Theorem 2. If a graph G with n vertices has a proper contact representation by unit cubes then it has at most $7n - \Omega(n^{2/3})$ edges, which is asymptotically tight.

Proof: Let Γ be any unit cube representation of G. By Theorem 1, each cube C of Γ has proper contact with at most 14 other cubes. This implies that the maximum number

of edges of G is at most $7n$. Moreover, in any unit cube representation of G there are $\Omega(n^{2/3})$ cube vertices that are in the "exterior" of Γ. These vertices cannot belong to cubes of degree 14. It follows that G has at most $7n - \Omega(n^{2/3})$ edges.

To prove that this upper bound is tight we pack $n = k^3$ (k odd) unit cubes in an axis-aligned bounding box B of volume n and side length k. Call the set of k^2 cubes with z-coordinate in $[i, i+1]$ L_i ($i = 1..k$). For every even i, shift L_i (for i even) by $(\epsilon, \epsilon, 0)$ (with $0 < \epsilon < 1$) so that each cube of L_i, except for those originally on the border of B, touches 4 cubes of L_{i-1} and L_{i+1}. Let R_{ij} be the row of k cubes in L_i with y-coordinate in $[j, j+1]$ ($j = 1..k$). For every even i and j, shift $R_{i,j}$ by $(\epsilon, 0, 0)$ so that each cube of $R_{i,j}$, except for those originally on the border of B, touches two cubes of $R_{i,j-1}$ and $R_{i,j+1}$. Now every cube, except for the $O(n^{2/3})$ cubes originally on the border of B, has degree 14. ∎

An immediate consequence of Theorem 1 is that any graph having at least one vertex of degree larger than 14 does not admit a proper contact representation by unit cubes. In addition, volume constraints imply that relevant families of graphs whose maximum vertex degree is smaller than 14 may not have a proper contact representation by unit cubes. We give an example of one such families below.

Lemma 1. Every complete binary tree whose height is larger than or equal to 14 does not admit a proper contact representation by unit cubes.

Proof: Let T be a complete binary tree with $n = 2^{h+1} - 1$ vertices and height h. The diameter of T is $2h$. Since T is connected, a proper contact representation of T by unit cubes has a bounding box with side lengths at most $2h + 1$ and thus volume at most $(2h + 1)^3$. On the other hand, the volume must be at least n. For $h \geq 14$, this results in a contradiction. ∎

Note that the argument of the lemma above implies that any graph that contains a vertex v such that the number of vertices with graph-theoretic distance at most d from v is greater than $(2d + 1)^3$ does not have a proper contact representation by unit cubes. Still, there are non-trivial classes of planar graphs that do have a proper contact representation by unit cubes, as the next theorem shows.

Theorem 3. Triangular grid, square grid, hexagonal grid, and parabolic grid graphs have a proper contact representation by unit cubes.

Proof Sketch: It is easy to formally prove that all the regular grids above have such representations. In fact, in these representations all the cubes are co-planar so providing (x, y)-coordinates for each cube suffices. We leave the details out and instead include a proof-by-picture, illustrating the regular tiling pattern (infinitely extendable) for the claimed grid graphs; see Fig. 4. ∎

3 Complexity of the Unit Cube Representability Problem

Here we give a compressed proof that it is NP-complete to determine whether a graph has a proper contact representation by unit cubes. Section 3.1 gives a "logic engine" [7] design, which is essentially a representation of a certain contact graph by unit cubes, such that *if* an instance of NAE 3-SAT is satisfiable, then the graph can be represented.
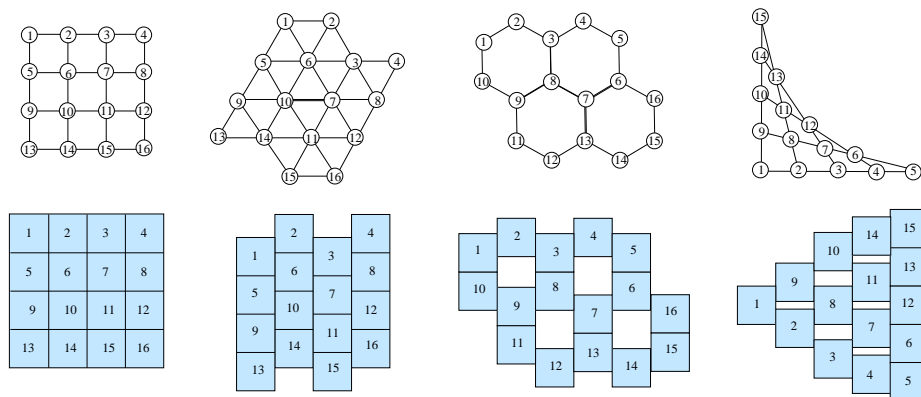
Fig. 4. Regular tiling patterns with unit cubes.

Section 3.2 uses the results of Section 2 to give tools for proving that the graph can be represented *only if* the NAE instance is satisfiable. Section 3.3 completes the proof.

3.1 Logic engine design

For an instance of NAE 3-SAT with m clauses and n variables, our logic engine consists of n *poles* (one for each variable and its complement) which rotate around a central *shaft*. The shaft is attached at each end to a *shell* which encases the entire assembly. The shaft passes through a *pole block* at the center of each pole. The uncomplemented side of each pole represents an uncomplemented variable, and the other side, the complement of that variable. Each side of a pole contains m *link blocks*. A *flag block* is attached to the ith link block from the jth pole block, on the uncomplemented side, if and only if i^{th} clause fails to contain the jth variable (uncomplemented); similarly, a flag block is attached to the ith link block on the complemented side of the jth pole if and only if the ith clause does not contain the jth variable in complemented form. Along the poles and the shaft, *spacer blocks* separate link and pole blocks. A schematic cross-section of the logic engine for an $m = n = 3$ instance is in Fig. 5. All blocks are $a \times a \times a$ cubes, where a is odd so that each block face has a unit cube face at its center. Unit cubes appear larger than scale. The side walls of each block are 2 units thick, as are the walls of the shell, which is a hollow rectangular box.

Each unit cube depicted in Fig. 5 makes face contact on two sides with a unit cube in the center of the face of a neighboring block (or shell); this allows poles and flagged link blocks to "turn" independently.

A variable is regarded as "true" if the uncomplemented part of the pole is positioned above the shaft. The flags can be positioned so that they do not intersect other flags or the shell if and only if there is at least one unflagged block in each row of blocks above and below the shaft, in which case all the remaining flags in the row can be rotated to point to the unflagged block. This corresponds to having at least one true
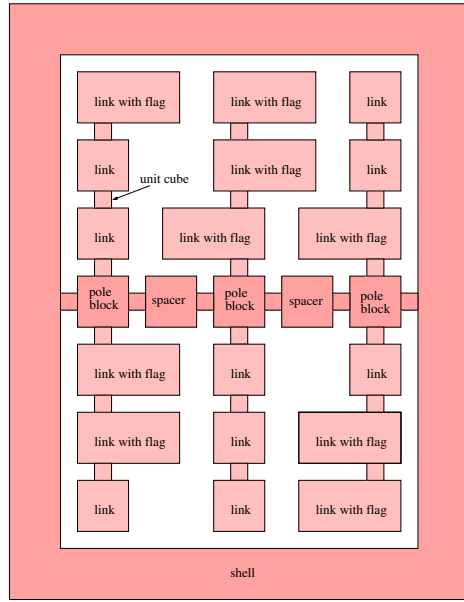
Fig. 5. Schematic cross-section of the logic engine (shaft and shell are darkly shaded).

literal and at least one false literal in each clause. Fig. 5 encodes a "yes" instance of $(x_1 \vee x_2 \vee \bar{x_3}) \wedge (x_1 \vee x_2 \vee x_3) \wedge (\bar{x_1} \vee x_2 \vee x_3)$, where $x_1 = T$, $x_2 = F$, and $x_3 = T$.

3.2 Making the blocks and shell rigid

The inner dimensions of the blocks and enclosing shell must accommodate unit cubes from additional graph structure needed to force the desired rigidity of these parts. The need for this extra, rigidifying structure is indicated by Fig. 6.
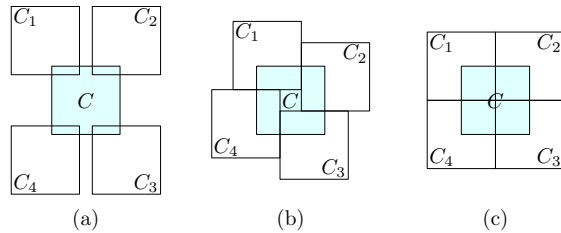

Fig. 6. Top view of 4 unit cubes on an extremal layer of a degree 14 unit cube C; the contact subgraph can be: (a) an independent set of size 4; (b) a chordless 4-cycle, with cubes unaligned; (c) a chordless 4-cycle, with aligned, tight cubes in a 2x2x1 box.

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A *cube graph* is the skeleton of a cube, i.e., a matched pair of chordless 4-cycles. Since the walls of the shell and the blocks have thickness 2, their associated graphs of desired contacts are composed of cube graphs. It is not hard to show that if for some reason, the 8 vertices in any cube subgraph in a block or shell must be realized *tightly*, i.e., by unit cubes with a $2 \times 2 \times 2$ bounding box, then the entire block or shell must be represented in the desired aligned, rigid way. Our proof defines and uses *crystal lattice graphs*, inspired by crystallography (see, e.g., [2]).
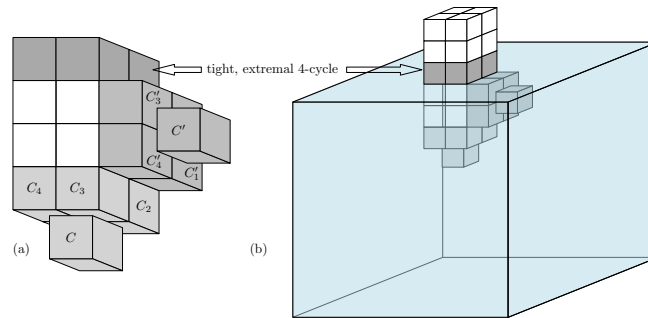

Fig. 7. (a) A portion of the seed graph. C and C' represent vertices of degree 14. C has an "top" cycle of neighbors C_1, C_2, C_3, C_4. C' has a "top" cycle of neighbors C'_1, C'_2, C'_3, C'_4. The remaining neighbors of C and C' are not shown. (b) The crystallizer graph, with a $10 \times 10 \times 10$ box enclosing all but the tight, extremal 4-cycle of its seed graph. Only a portion of the seed graph is shown.

To make a cube subgraph *rigid*, i.e., to force it, and hence the shell or block graph to which it belongs, to have a rigid, unique, aligned representation, we attach the cube subgraph to a *crystallizer* graph, which in turn contains a *seed* graph. To define these two special graphs, we start with the particular contact graph of neighbors shown in Fig. 3. This neighborhood contact graph uniquely identifies the faces of degree 14 cube C and gives unique identities to its neighbors, both in the contact graph and in its representation A by unit cubes. For example, we say the unique chordless 4-cycle that must be represented by cubes lying in an extremal layer of A lies in the *top* layer of A, in contact with the *top* face of C. We call the *left* face of C the one in contact with the unique pair of middle layer cubes that make face contact with each other and edge contacts with C. We say the unique pair of cubes in the bottom layer that contact each other also contact the bottom face of C at the *front*; see Fig. 3.

To create the *seed graph*, we imagine taking the assemblies of two degree 14 vertices and their neighborhoods, as described above, and attaching the top of each assembly to a tight representation of a central cube graph, as shown in Fig. 7. The labeled 4-cycles of cubes must be represented tightly, with no hole as in Fig. 7 (b), because the face planes of the top layers align the cubes. Finally, we match an additional chordless 4-cycle to the chordless 4-cycle in the central cube graph, as shown at the upper part of Fig. 7 (a), to obtain a *seed graph* G_1. The *crystallizer graph* is simply a cube graph

with the vertices of one of its chordless 4-cycles identified with the vertices of the tight, extremal chordless 4-cycle of the seed graph; see Fig. 7 (b). This cube graph subgraph of the crystallizer graph is always tight in any crystallizer graph representation, and hence can force rigid alignment of representations of crystal lattice graphs such as the graphs for the blocks and the shell when it is identified with one of their cube subgraphs.

3.3 Recognizing proper contact graphs of unit cubes is NP-complete

To prove membership in NP, is suffices to prove that if a representation exists, then one with a polynomially bounded description exists; this can be done by a technique similar to that of the proof of Theorem 3.2 of Czyzowicz et al. [4].

Any instance of NAE-3SAT can be transformed in polynomial time to a logic engine graph made of crystal lattice graphs for the shell, link, flag, pole, and spacer blocks together with the crystallizer graphs that make these parts rigid, together with the vertices whose representations allow the poles and links to turn independently (i.e., to be spatially positioned in alternative ways). If the NAE-3SAT instance can be satisfied, then the logic engine graph can be properly realized in at least one way (see Section 3.1). On the other hand, if the logic engine graph can be represented, then its crystallizer subgraphs force the parts to be drawn rigidly and aligned; the shell forces the poles, links and flags into a flattened configuration, such that an unflagged position must appear in each row above and below the shaft to avoid intersection of unit cubes. Thus each clause has at least one true and one false literal. This completes our compressed proof. ∎

4 Cubes of Varying Size

Motivated by the result of the previous section, we consider graphs that admit a proper contact representation by cubes of varying size. We start by establishing an upper bound on the number of edges that such graphs can have. The proof is similar to that of Theorem 1 and it is sketched in the appendix.

Theorem 4. If a graph G with n vertices has a proper contact representation by cubes then it has at most $14n$ edges.

We now prove that a fairly general family of planar graphs, namely the class of partial planar 3-trees, has a proper contact representation by cubes. We recall that a partial planar 3-tree is a subgraph of an Apollonian network or, equivalently, a planar graph with tree-width at most 3. For example, outerplanar graphs and series-parallel graphs are partial planar 3-trees. It may be worth noticing that partial planar 3-trees with n vertices can have vertices of degree $\Omega(n)$; hence, by Theorem 1 they do not have a proper contact representation by unit cubes. A complete proof of the following theorem is given in the appendix.

Theorem 5. Every partial planar 3-tree has a proper contact representations by cubes.

Proof Sketch: We first show that every maximal planar 3-tree admits a proper contact representation by cubes. We use the fact that every maximal planar 3-tree can be obtained starting from K_4 and adding a vertex of degree 3 inside a triangular face and

connecting it to all vertices of the face. We represent K_3 with three cubes that touch one another so that they form a corner. The fourth vertex is represented by a smaller cube that fits in the corner and creates three new corners, each corresponding to the three inner triangular faces of K_4. We now construct the representation by adding a vertex at a time. Every time a vertex of degree 3 is inserted, it is represented by a smaller cube that fits in the appropriate corner, creating 3 more corners.

Moreover, we can also represent partial 3 trees. We sketch the main idea next. Any planar partial 3-tree G can be augmented to a maximal planar 3 tree G' by adding edges. Let Γ' be a proper contact representation by cubes of G', computed as described above. Let (u, v) be a vertex of G' that is not in G. Edge (u, v) is represented in Γ' by a proper contact between the cube C_u representing u and the cube C_v representing v. Also, assume that v was added to the representation after u had already been drawn, which corresponds to the fact that v is inserted inside a face having u on its boundary. By the drawing procedure sketched above, cube C_v is smaller in size than C_u and it properly touches C_u orthogonally to either the x-, or the y-, or the z-direction. Assume, for concreteness, that C_u and C_v touch with a plane orthogonal to the positive x-direction. Let $C_v, C_1, C_2, \ldots, C_k$ be all cubes that can be stabbed by a ray originating in C_u and shooting in the positive x-direction. In order to delete edge (u, v), we compute a small enough real positive value ϵ such that if C_v is moved along the positive x-direction by ϵ none of the proper contacts between $C_1, C_2, \ldots, C_k$ and the remaining cubes of Γ' are destroyed. ∎

5   Proper Contact Representations by Cuboids

In this section we further relax the constraint on the polyhedra representing the vertices and consider proper contact representations of graphs where vertices are represented by cuboids. We present two new (simple) proofs of the following result that was first established by Thomassen [16].

Theorem 6. Any planar graph has a proper contact representation by cuboids.

The idea behind the first proof is to use the concept of *canonical ordering* of a plane triangulation, introduced by de Fraysseix, Pach, and Pollack in [6]. We add vertices in the canonical order and maintain a staircase rectilinear profile in 2D of the "active" vertices (that is, the vertices on the outerface of G_i), at the same z level. The "height" of the realization (the 3rd dimension) corresponds to the index of the current vertex that we process. When a new vertex v_i is added it connects to a chain of vertices $v_l, v_{l+1}, \ldots, v_r$ of length 2 or more. The contact representation of G_{i-1} has height $z = i - 1$. The new vertex v_i is represented by a cuboid that covers (is placed on top) of the cuboids representing $v_{l+1}, \ldots, v_{r-1}$ and has proper contact with the cuboids representing v_l and v_r on surfaces that are orthogonal to the x- and to the y-direction. Then all new active cuboids are extended one unit in the z dimension. In particular, the covered vertices do not grow, while all others gain one unit in the z-direction. See the appendix for a detailed description of this proof.

The second proof uses Schnyder's realizers. We start by adding dummy edges to G so that G is a maximal planar graph. Later, we will avoid contacts that represent dummy edges in G by slightly shrinking the dimensions of some cuboids. Let a_1, a_2, a_3 be the

exterior face of an embedding of G. A *Schnyder realizer* [15] of G is a partition of the interior edges of G into three sets T_1, T_2, T_3 of directed edges such that for each interior vertex v: (1) v has outdegree one in each of T_1, T_2, and T_3, and (2) the edges around v in counterclockwise order are leaving in T_1, entering in T_3, leaving in T_2, entering in T_1, leaving in T_3, entering in T_2.

We may assume that the directed edges in T_i form a tree rooted at external vertex a_i. We add the edges (a_2, a_1) to T_1, (a_3, a_2) to T_2, and (a_1, a_3) to T_3; see Fig. 8.

Let $<_X, <_Y, <_Z$ be a set of three total orders on the vertices of G with $<_X$ consistent with T_1, T_2^{-1}, T_3^{-1}, $<_Y$ consistent with T_1^{-1}, T_2, T_3^{-1}, and $<_Z$ consistent with T_1^{-1}, T_2^{-1}, T_3. Here, T_i^{-1} is the transpose of T_i. The sequence $(<_X, <_Y, <_Z)$ exists and is called a 3-dimensional representation of G [14].

We use these three total orders to define our proper contact representation of G by cuboids. For vertex v, let $x_M(v)$, $y_M(v)$, and $z_M(v)$ be the rank of v in the *inverse* of $<_X, <_Y$, and $<_Z$ respectively. (Using the inverse makes the pictures similar to T-contact systems [5].) For $(u, v) \in T_1$, let $x_m(u) = x_M(v)$. Let $x_m(a_1) = x_m(a_3) = 0$. Since every internal vertex u and a_2 has outdegree one in T_1, this defines $x_m(u)$ for all vertices. If (u, v) is a dummy edge then subtract some ϵ ($0 < \epsilon < 1$) from $x_m(u)$. For $(u, v) \in T_2$, $y_m(u) = y_M(v)$ (minus ϵ if a dummy edge) and $y_m(a_1) = y_m(a_2) = 0$. For $(u, v) \in T_3$, $z_m(u) = z_M(v)$ (minus ϵ if a dummy edge) and $z_m(a_2) = z_m(a_3) = 0$.

For a vertex v, define the v-cuboid as the region $([x_m(v), x_M(v)], [y_m(v), y_M(v)], [x_m(v), z_M(v)])$. Let $B(<_X, <_Y, <_Z)$ be the set of v-cuboids defined by $<_X, <_Y$, and $<_Z$ (obtained from the Schnyder realizer (T_1, T_2, T_3) of G); see Fig. 8. The proof that $B(<_X, <_Y, <_Z)$ is a proper contact representation of G by cuboids can be found in the appendix.
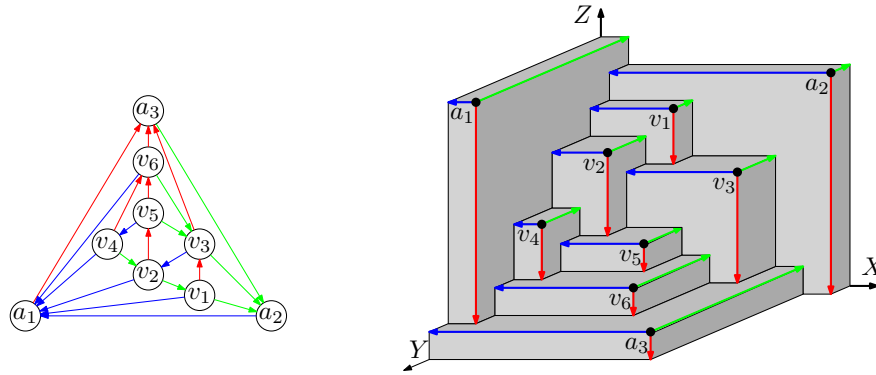

Fig. 8. A Schnyder realizer with $<_X$: $a_2 a_3 v_3 v_6 v_5 v_1 v_2 v_4 a_1$, $<_Y$: $a_3 a_1 v_6 v_4 v_5 v_2 v_3 v_1 a_2$, and $<_Z$: $a_1 a_2 v_1 v_2 v_3 v_4 v_5 v_6 a_3$ and the corresponding proper contact representation by cuboids. The arrangement of cuboids may enclose regions of empty space.

6 Open Problems

1. Does every planar graph admit a proper contact representation by cubes?

2. Can we characterize/recognize the class of graphs that have a proper contact representation by cubes (e.g., K_5 is not representable but $K_{3,3}$ is)?

3. What is the complexity of deciding whether a planar graph has a proper contact representation by unit cubes (we have shown that the problem is NP-complete for general graphs)?

4. The question above is interesting also for just binary trees.

7 Acknowledgments

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Appendix

Proof of Theorem 4

Theorem 4. If a graph G with n vertices has a proper contact representation by cubes then it has at most $14n$ edges.

Proof Sketch: Let Γ be any cube representation of G. Consider a cube C having smallest dimension in Γ. By Theorem 1, there can be at most 14 cubes touching C and having size larger than or equal to the one of C. If we remove C from Γ, we delete at most 14 edges. We remove all edges of Γ by repeatedly removing a cube of smallest size. Since there are at most n cubes and each deletion removes at most 14 edges, we have that the number of edges in G is at most $14n$. ∎

Proof of Theorem 5

Theorem 5. Every partial planar 3-tree has a proper contact representation by cubes.

Proof: Augment G to a maximal planar 3-tree G' by adding dummy edges to it. The edges of G' are hence partitioned into *real edges*, that belong both to G and to G', and *dummy edges*, that belong to G' and not to G.

We first show that G' has a proper contact representation by cubes Γ'. We later modify Γ' to obtain a proper contact representation of G by cubes.

In order to construct Γ', we use the fact that every n-vertex maximal planar 3-tree is a maximal planar graph with tree-width 3, hence it can be constructed starting from K_3 and repeatedly adding a vertex of degree 3 inside a triangular face of a planar embedding of a previously constructed maximal planar 3-tree with one less vertex. Such a construction determines an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G', where v_1, v_2, and v_3 are the vertices of the starting K_3. For each $k \geq 3$, denote by G'_k the subgraph of G' induced by $v_1, v_2, \ldots, v_k$. We construct Γ' starting from a representation of (v_1, v_2, v_3) and drawing the vertices in G' one at a time, in order $v_4, v_5, \ldots, v_n$. For each $k \geq 3$, the constructed proper contact representation by cubes Γ'_k of G'_k satisfies the following invariants:

- *Invariant A:* For each v_i ($4 \leq i \leq k$), there exists a labeling of its adjacent vertices $a_i, b_i, c_i \in G'_{i-1}$ such that the bottom (back, left) side of $C(v_i)$ touches only the cube $C(a_i)$ ($C(b_i)$, $C(c_i)$, respectively).

- *Invariant B:* For each face f of G'_k, there exists a labeling of its vertices a, b, c and a point p_f in Γ'_k such that, for every $\ell > 0$, the cube C with $x_m(C) = x(p_f)$, $y_m(C) = y(p_f)$, $z_m(C) = z(p_f)$, and with side length ℓ touches only only the cube $C(a)$ ($C(b)$, $C(c)$) along its bottom (back, left, respectively) side.

When $k = 3$, a proper contact representation by cubes satisfying Invariants A and B is constructed by drawing v_1, v_2, and v_3 as three cubes of side length 2, with $x_m(v_1) = y_m(v_1) = -1$, $z_m(v_1) = -2$, $x_m(v_2) = -2$, $y_m(v_2) = -1$, $z_m(v_1) = 0$, $x_m(v_3) = 0$, $y_m(v_3) = -2$, and $z_m(v_3) = 0$. Observe that Invariant B is satisfied with $p_f \equiv (0, 0, 0)$; see Fig. 9(a).

Assume now that $k > 3$ and let Γ'_{k-1} be a a proper contact representation of G'_{k-1} by cubes satisfying Invariants A and B. Let $f = (a_k, b_k, c_k)$ be the face of G'_{k-1} into which v_k is inserted to form G'_k. Let $l > 0$ be any real number such that
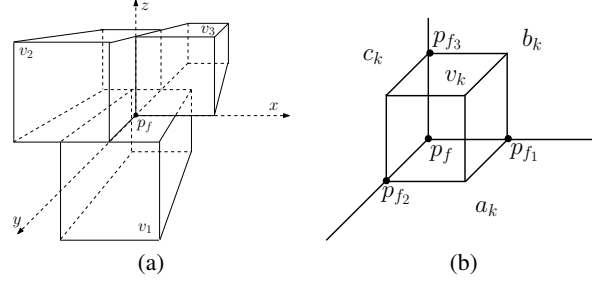
Fig. 9. Illustration of the algorithm for constructing a proper contact representation of G' by cubes. (a) Construction of Γ'_3. (b) Insertion of vertex v_k in Γ'_{k-1}.

$l < \min\{x_M(C(a_k)) - x(p_f), y_M(C(a_k)) - y(p_f), x_M(C(b_k)) - x(p_f), z_M(C(b_k)) - z(p_f), y_M(C(c_k)) - y(p_f), z_M(C(c_k)) - z(p_f)\}$. Construct Γ'_k from Γ'_{k-1} by drawing v_k so that $x_m(v_k) = x(p_f)$, $y_m(v_k) = y(p_f)$, $z_m(v_k) = z(p_f)$, and the side length of $C(v_k)$ is l; see Fig. 9(b). Since Γ'_{k-1} satisfies Invariant B, it follows that $C(v_k)$ touches $C(a_k), C(b_k)$, and $C(c_k)$ along its bottom, back, and left side, respectively, and touches no cube other than $C(a_k), C(b_k)$, and $C(c_k)$; hence Γ'_k is a proper contact representation of G'_k by cubes satisfying Invariant A. We prove that Invariant B is satisfied by Γ'_k. For each face f of G'_k which is not incident to v_k, point p_f has the same coordinates as in Γ'_{k-1}. Moreover, denoting by f_1 face (a_k, b_k, v_k), by f_2 face (a_k, v_k, c_k), and by f_3 face (v_k, b_k, c_k) of G'_k, we have that points $p_{f_1} \equiv (x(p_f) + l, y(p_f), z(p_f))$, $p_{f_2} \equiv (x(p_f), y(p_f) + l, z(p_f))$, and $p_{f_3} \equiv (x(p_f), y(p_f), z(p_f) + l)$ satisfy Invariant B for the faces f_1, f_2, and f_3, respectively. Namely, for every $\ell > 0$ the cube C with $x_m(C) = x(p_{f_1})$, $y_m(C) = y(p_{f_1})$, $z_m(C) = z(p_{f_1})$, and with side length ℓ touches $C(a_k), C(b_k)$, and $C(v_k)$ along its bottom, back, and left side, respectively, by construction; moreover, since C is contained inside the cube C' with side length $\ell + l$ and with $x_m(C') = x(p_f)$, $y_m(C') = y(p_f)$, $z_m(C') = z(p_f)$ and since Γ_{k-1} satisfies Invariant B, we have that C touches no other cube representing a vertex of G'_k. Analogous considerations hold for the cubes associated with faces f_2 and f_3.

Next, we show how to remove dummy edges of G' from Γ', thus obtaining a proper contact representation by cubes Γ of G. Define v_k to be the *x-child* of c_k, the *y-child* of b_k, and the *z-child* of a_k. Define the *x-descendants* of u as follows: If u has no x-child, then it has no x-descendants; if u has an x-child, v, then the x-descendants of u are v and its x-descendants; y- and z-descendants are defined analogously. Consider again the vertices of G'_k in order $v_4, v_5, \ldots, v_n$. When vertex v_k is considered, edges $(v_k, a_k), (v_k, b_k)$, and (v_k, c_k) are removed, if they are dummy. This is done as follows. Suppose that (v_k, a_k) is a dummy edge (we can deal with edges (v_k, b_k) and (v_k, c_k) analogously). Let $\epsilon > 0$ be a real number such that, if the z-coordinates of the vertices of $C(v_k)$ and of the vertices of the cubes representing its z-descendants are increased by ϵ, two cubes touch along a plane parallel to $x = 0$ or $y = 0$ after the modification if and only if they touched before the modification. Observe that, since Γ' is a proper contact representation by cubes, such an ϵ always exists. Then, increase by ϵ the z-coordinates of the vertices of $C(v_k)$ and of the vertices of the cubes representing its

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z-descendants. By Invariant A, $C(v_k)$ touches $C(u_k)$ along its bottom side and touches no cube other than $C(u_k)$ along its bottom side. Hence, the performed modification does not alter the adjacency between any two vertices in Γ', except for v_k and a_k, that are not adjacent any longer in Γ'. Finally, dummy edges between v_1, v_2, and v_3 can be handled analogously. ∎

Using canonical ordering to prove Theorem 6

Let G be any planar graph. Augment G to a maximal planar graph G' by adding dummy edges to it. The edges of G' are hence partitioned into *real edges*, that belong both to G and to G', and *dummy edges*, that belong to G' and not to G.

Lemma 2. *[6]* Let G' be a maximal planar graph embedded in the plane with exterior face (u, v, w). Then, there exists a labeling of the vertices $v_1 = u, v_2 = v, v_3, \ldots, v_n = w$ meeting the following requirements for every $4 \le k \le n$.

1. The subgraph $G'_{k-1} \subset G'$ induced by $v_1, v_2, \ldots, v_{k-1}$ is 2-connected, and the boundary of its exterior face is a cycle C'_{k-1} containing the edge (u, v);
2. v_k is in the exterior face of G'_{k-1}, and its neighbors in G'_{k-1} form an (at least 2-element) sub-interval of the path $C'_k - (u, v)$.

Denote by G_k the graph G'_k restricted to the real edges of G'. Observe that $G_n = G$. Further, denote by $x_m(t)$, $x_M(t)$, $y_m(t)$, $y_M(t)$, $z_m(t)$, and $z_M(t)$ the minimum x-coordinate, the maximum x-coordinate, the minimum y-coordinate, the maximum y-coordinate, the minimum z-coordinate, and the maximum z-coordinate of a cuboid representing a vertex t of G_k, respectively. Our algorithm constructs, for each $3 \le k \le n$, a proper contact representation by cuboids Γ_k of G_k such that the following invariants are satisfied.

- *Invariant 1:* The border of Γ_k is *z-free*, i.e., the z-coordinates of the top side of each cuboid representing a vertex in C'_k are equal and can be arbitrarily and independently increased still maintaining the property that Γ_k is a proper contact representation of G_k by cuboids.
- *Invariant 2:* The border of Γ_k is an (x, y)-*staircase*, i.e., it holds $x_M(w_i) < x_M(w_{i+1})$ and $y_M(w_i) > y_M(w_{i+1})$, for each $1 \le i \le m - 1$.

Our algorithm works by induction on k. For $k = 3$, a proper contact representation by cuboids satisfying Invariants 1 and 2 can be constructed by assigning coordinates to v_1, v_2, and v_3 as follows: Set $z_m(v_1) = z_m(v_2) = z_m(v_3) = 0$, $z_M(v_1) = z_M(v_2) = z_M(v_3) = 1$, $x_m(v_1) = 0$, $x_M(v_1) = 1$, $y_m(v_1) = 0$, $y_M(v_1) = 4$, $x_M(v_2) = 4$, $y_m(v_2) = 0$, $y_M(v_2) = 1$, $x_M(v_3) = 3$, and $y_M(v_3) = 3$. Further, if edge (v_1, v_2) is real, then set $x_m(v_2) = 1$, otherwise set $x_m(v_2) = 2$; if edge (v_1, v_3) is real, then set $x_m(v_3) = 1$, otherwise $x_m(v_3) = 2$; finally, if edge (v_2, v_3) is real, then set $y_m(v_3) = 1$, otherwise set $y_m(v_3) = 2$; see Fig. 10.

Now suppose that a proper contact representation by cuboids Γ_k satisfying Invariants 1 and 2 can be constructed for G_k, for some $3 \le k \le n - 1$. We show how to construct proper contact representation by cuboids Γ_{k+1} of G_{k+1} satisfying Invariants 1 and 2. Denote by $w'_1, w'_2, \ldots, w'_{p-1}, w'_p$ the neighbors of v_{k+1} in G'_k. Let Z be the real number such that the top sides of the cubes representing $w'_1, w'_2, \ldots, w'_{p-1}, w'_p$
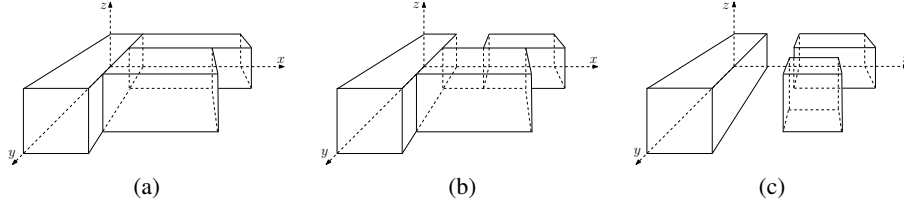
Fig. 10. Three of the eight base cases of the algorithm. (a) Edges (v_1, v_2), (v_1, v_3), and (v_2, v_3) are all real. (b) Edges (v_1, v_3) and (v_2, v_3) are real, while edge (v_1, v_2) is dummy. (c) Edges (v_1, v_2), (v_1, v_3), and (v_2, v_3) are all dummy.

lie on the plane $z = Z$ (observe that all such top sides lie on the same plane by Invariant 1). Let Z' and Z'' be any real numbers such that $Z'' > Z' > Z$. For each vertex w'_i in $w'_2, \ldots, w'_{p-1}$ such that edge (w'_i, v_{k+1}) is real, increase to Z' the z-coordinate of the top side of the cuboid representing w'_i in Γ_k; for each vertex that is in C'_k and that is not in $w'_2, \ldots, w'_{p-1}$, increase to Z'' the z-coordinate of the top side of the cuboid representing such a vertex in Γ_k. By Invariant 1, the modified Γ_k is still a proper contact representation of G_k by cuboids. Set $z_m(v_{k+1}) = Z'$ and $z_M(v_{k+1}) = Z''$. Let X and Y be any real numbers such that $x_M(w'_{p-1}), x_m(w'_p) < X < x_M(w'_p)$ and $y_M(w'_2), y_m(w'_1) < Y < y_M(w'_1)$, respectively. Further, let X' and Y' be real numbers such that $x_M(w'_1) < X'$, such that $y_M(w'_p) < Y'$, and such that, for each $2 \le i \le p-1$, $Y' < y_M(w'_i)$ and $X' < x_M(w'_i)$. Set $x_M(v_{k+1}) = X$ and $y_M(v_{k+1}) = Y$. Further, if edge (w'_1, v_{k+1}) is real, then set $x_m(v_{k+1}) = x_M(w'_1)$, otherwise set $x_m(v_{k+1}) = X'$. Finally, if edge (w'_p, v_{k+1}) is real, then set $y_m(v_{k+1}) = y_M(w'_p)$, otherwise set $y_m(v_{k+1}) = Y'$; see Fig. 11.
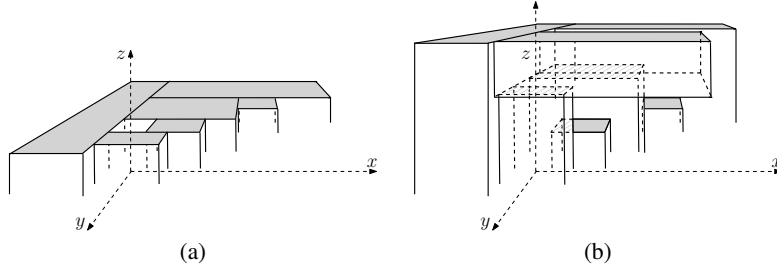

Fig. 11. Inductive case of the algorithm. (a) Γ_k (in which only the top sides of the cuboids representing the vertices $w_1, w_2, \ldots, w_6$ in C'_k are shown). (b) Γ_{k+1} (in which only the cuboid representing v_{k+1} and the top sides of the cuboids representing vertices in C'_k are shown). In this example v_{k+1} is adjacent to each of $w_1, w_2, \ldots, w_6$ in G'_{k+1}, and is adjacent to w_1, w_2, and w_4 in G_{k+1}.

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Proof of Lemma 3

Lemma 3. $B(<_X, <_Y, <_Z)$ is a proper contact representation of G by cuboids.

Proof: The upper x-normal face of the a_1-cuboid touches the lower x-normal face of the a_2-cuboid. Similarly, the a_2-cuboid and the a_3-cuboid touch y-normal faces. and the a_3-cuboid and a_1-cuboid touch z-normal faces. Thus the exterior edges of G are represented.

Consider an interior edge (u, v) in G. Assume T_3 contains the directed edge (u, v). The argument is similar for T_1 and T_2. We show that the corner $(x_M(u), y_M(u), z_m(u))$ of the u-cuboid lies in the interior of the face $([x_m(v), x_M(v)], [y_m(v), y_M(v)], z_M(v))$ of the v-cuboid. By construction, $z_m(u) = z_M(v)$. Since $(u, v) \in T_3$, $v <_X u$ and $v <_Y u$ thus $x_M(u) < x_M(v)$ and $y_M(u) < y_M(v)$.

If v is an external vertex then $v = a_3$ so $x_M(u) > x_m(v) = 0$ and $y_M(u) > y_m(v) = 1$.

Otherwise, to establish $x_M(u) > x_m(v)$, let (v, w) be the edge from v in T_1. By Property (2) of Schnyder realizers, uvw forms a triangle in G. Since $<_X$, $<_Y$, and $<_Z$ are acyclic, either $(u, w) \in T_1$ or $(w, u) \in T_2$. In either case, $u <_X w$, thus $x_M(u) > x_M(w) = x_m(v)$. A similar argument can be used to show that $y_M(u) > y_m(v)$. Thus the lower z-normal face of the u-cuboid touches the upper z-normal face of the v-cuboid. ∎