Threshold-Coloring and Unit-Cube Contact Representation of Graphs

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Abstract. In this paper we study *threshold coloring* of graphs, where the vertex colors represented by integers are used to describe any spanning subgraph of the given graph as follows. Pairs of vertices with near colors imply the edge between them is present and pairs of vertices with far colors imply the edge is absent. Not all planar graphs are threshold-colorable, but several subclasses, such as trees, some planar grids, and planar graphs without short cycles can always be threshold-colored. Using these results we obtain unit-cube contact representation of several subclasses of planar graphs. Variants of the threshold coloring problem are related to well-known graph coloring and other graph-theoretic problems. Using these relations we show the NP-completeness for two of these variants, and describe a polynomial-time algorithm for another.

1 Introduction

Graph coloring is among the fundamental problems in graph theory. Typical applications of the problem and its generalizations are in job scheduling, channel assignments in wireless networks, register allocation in compiler optimization and many others [17]. In this paper[5] we consider a new graph coloring problem in which we assign colors (integers) to the vertices of a graph G in order to define a spanning subgraph H of G. In particular, we color the vertices of G so that for each edge of H, the two endpoints are near, i.e., their distance is within a given "threshold", and for each edge of $G \setminus H$, the endpoints are far, i.e., their distance greater than the threshold; see Fig 1.

The motivation of the problem is twofold. First, such coloring can be used for the Frequency Assignment Problem [11], which asks for assigning frequencies to transmitters in radio networks so that only specified pairs of transmitters can communicate with each other. Second, such coloring can be used in the context of the geometric problem of *unit-cube contact representation* of planar graphs [3]. Here we show how to use a threshold coloring of a planar graph in order to obtain such unit-cube contact representation. Suppose a planar graph G has a unit-cube contact representation where one face of each cube is co-planar; see Fig. 1(a). Assume that we can define a spanning subgraph H of G by our particular vertex coloring. We show that it is possible to compute a unit-cube contact representation of H by lifting the cube for each vertex v by the amount equal to the color of v (where the size or side-length of the cubes are roughly equal to the threshold); see Fig. 1(b).

[5] A full version of this paper is on ArXiv [1].

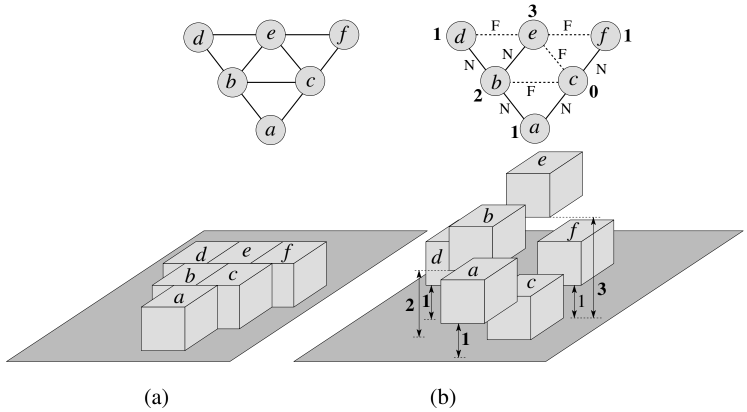


Fig. 1. (a) A planar graph G and its unit-cube contact representation where the bottom faces of all cubes are co-planar, (b) a spanning subgraph H of G with a $(4, 1)$-threshold-coloring and its unit-cube contact representation. Far edges are shown dashed, near edges are shown solid.

1.1 Problem Definition

An *edge-labeling* of graph $G = (V, E)$ is a mapping $\ell : E \rightarrow \{N, F\}$ assigning labels N or F to each edge and the pair $\{N, F\}$ defines a partition of the edges into *near* and *far* edges. Let $r \geq 1$ and $t \geq 0$ be two integers and let $[1 \ldots r]$ denote a set of r consecutive integers. For a graph $G = (V, E)$ and an edge-labeling $\ell : E \rightarrow \{N, F\}$, a (r, t)*-threshold-coloring* of G with respect to ℓ is a *coloring* $c : V \rightarrow [1 \ldots r]$ such that for each edge $e = (u, v) \in E$, $e \in N$ if and only if $|c(u) - c(v)| \leq t$. We call r and t the *range* and the *threshold*. Note that the set of near edges defines a spanning subgraph $H = (V, N)$ of G, where H is a *spanning subgraph* of graph G if it contains all vertices of G. H is a *threshold subgraph* of G if there exists such a threshold-coloring.

A graph G is *total-threshold-colorable* if for every edge-labeling ℓ of G there exists an (r, t)-threshold-coloring of G with respect to ℓ for some $r \geq 1, t \geq 0$ (for every partition of edges of G into near and far edges, we can produce vertex colors so that endpoints of near edges receive near colors, and endpoints of far edges receive colors that are far apart). A graph G is (r, t)*-total-threshold-colorable* if it is total-threshold-colorable for the range r and threshold t. We consider the following problem variants:

Problem 1. (**Total-Threshold-Coloring Problem**) Given a graph G, is G total-threshold-colorable, that is, is every spanning subgraph of G a threshold subgraph of G?

The next problem asks if a particular spanning graph H of G is threshold-colorable.

Problem 2. (**Threshold-Coloring Problem**) Given a graph G and a spanning subgraph H, is H a threshold subgraph of G for some integers $r \geq 1, t \geq 0$?

The next problem assumes that G is the complete graph; then H is an *exact-threshold* graph if H is a threshold subgraph of the complete graph G for some $r \geq 1, t \geq 0$.

Problem 3. (**Exact-Threshold-Coloring Problem**) Given a graph H, is H an exact-threshold graph?

The final problem assumes that the threshold and the range are part of the input:

Problem 4. (**Fixed-Threshold-Coloring Problem**) Given a graph G, a spanning subgraph H, and integers $r \geq 1, t \geq 0$, is H (r, t)-threshold-colorable?

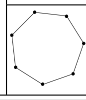
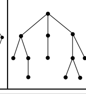
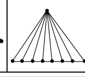
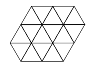
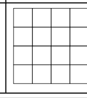
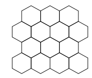
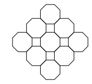
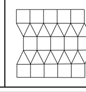
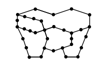
graph classes	Cycle	Tree	Fan	Triangular Grid	Square Grid	Hexagonal Grid	Octagonal-Square Grid	Square-Triangle Grid	Planar Graph w/o Cycles of size ≤ 9
									
threshold coloring	$r = 3$, $t = 0$	$r = 2$, $t = 0$	$r = 5$, $t = 1$	No	Open	$r = 5$, $t = 1$	$r = 5$, $t = 1$	No	$r = 8$, $t = 2$
unit-cube contact	Yes	No	No	Open	Yes	Yes	Yes	Open	No

Table 1. Results on the **Total-Threshold-Coloring Problem**. "No" entries in the last row follow from the fact that graphs with vertices of high degrees cannot have unit-cube representation [3].

1.2 Related Work

Many graph theoretic problems deal with vertex coloring a graph and numerous graph classes are defined based on such coloring; see [2] for a survey. To the best of our knowledge, total-threshold-colorability defines a new class of graphs. Here we mention two related classes: threshold and difference graphs. *Threshold graphs* are ones for which there is a real number S and for every vertex v there is a real weight a_v such that (v, w) is an edge if and only if $a_v + a_w \geq S$ [15]. A graph is a *difference graph* if there is a real number S and for every vertex v there is a real weight a_v such that $|a_v| < S$ and (v, w) is an edge if and only if $|a_v - a_w| \geq S$ [12]. Note that for both classes the threshold (real number S) defines edges between all pairs of vertices, while in our setting the threshold only defines edges of a (not necessarily complete) graph G.

Another related graph coloring problem is the *distance constrained graph labeling*. Here the goal is to find $L(p_1, \ldots, p_k)$-labeling of the vertices of a graph so that for every pair of vertices at distance at most $i \leq k$ we have that the difference of their labels is at least p_i. The most studied variant is $L(2, 1)$-labeling [7, 10]. In [10] it was shown that minimizing the number of labels in $L(2, 1)$-labelings is NP-complete, even for graphs with diameter 2. Further, Fiala *et al.* [6] proved that it is also NP-complete to determine whether a labeling exists with at most k labels for every fixed integer $k \geq 4$.

A threshold-coloring of a planar graph can be used to find a contact representation of the graph with cuboids (axis aligned boxes) in 3D. Thomassen [18] shows that any planar graph has a proper contact representation by cuboids in 3D. In a *contact representation* of a graph, the vertices are represented by cuboids (or other polygonal shapes) and the edges are realized by a common boundary of the two corresponding cuboids. A contact representation is *proper* if for each edge the corresponding common boundary has non-zero area. Felsner and Francis [5] prove that any planar graph has a (non-proper) contact representation by cubes. Bremner *et al.* [3] prove that the same result does not hold when using only unit cubes. Our results on threshold-coloring of planar graphs translates to results on classes of planar graphs that can be represented by contact of unit cubes.

1.3 Our Contributions

First we show some connections between threshold-coloring and other graph problems. Specifically, we show that the **Threshold-Coloring Problem** and the **Fixed-Threshold-Coloring Problem** are NP-complete by reductions from the proper interval graph sandwich problem and standard vertex coloring, and the **Exact-Threshold-**

Coloring Problem can be solved in linear time via equivalence to proper interval graph recognition. Next we study the **Total-Threshold-Coloring Problem** for various planar graph classes. Specifically, we show that trees, hexagonal grids, planar graphs without any cycles of length ≤ 9 are total-threshold-colorable, while the triangular grid and 4-3 grid are not; these results are summarized in Table 1. Finally we show how to use the threshold-coloring to compute unit-cube contact representations for several subclasses of planar graphs; see the last column of Table 1.

2 Threshold-Coloring and Other Graph Problems

We begin by showing the connections between threshold-colorability and some classical graph-theoretical and graph coloring problems.

2.1 Vertex Coloring Problem

Let $G = (V, E)$ be a graph. We call G k-vertex-colorable if there exists a coloring $c : V \rightarrow [1 \dots k]$ such that for any edge $(u, v) \in E$, $c(u) \neq c(v)$, that is, u and v have different colors. Given an input graph G and an integer $k > 0$, the vertex coloring problem asks whether there exists a k-vertex-coloring of G. Lemma 1 immediately follows from the definition.

Lemma 1. *Let $G = (V, E)$ be a graph and let k be a positive integer. Define an edge-labeling $\ell : E \rightarrow \{N, F\}$ that assigns each edge the label F, that is, for each edge $e \in E$, $\ell(e) = F$. Then G has a k-vertex-coloring if and only if there exists a $(k, 0)$-threshold-coloring of G with respect to ℓ.*

2.2 Proper Interval Representation Problem

An *interval representation* [2] for a graph $G = (V, E)$ is one where each vertex v of G is represented by an interval $I(v)$ of $\mathbb{R}$ such that for any edge $(u, v) \in E$, the intervals $I(u)$ and $I(v)$ have a non-empty intersection. A *proper interval graph* [2] is one that has an interval representation such that no interval properly contains another. Equivalently, a *proper interval graph* is one that has an interval representation with unit intervals [16]. The problem of proper interval representation for a graph G asks whether G has a proper interval representation. The problem has been studied extensively [4, 13, 14].

Lemma 2. *H is an exact-threshold graph if and only if it is a proper interval graph.*

Proof. (sketch) If a graph $H = (V, E)$ is an exact-threshold graph, there are integers $r \geq 1, t \geq 0$ and a mapping $c : V \rightarrow [r]$ such that for any pair $u, v \in V$, $|c(u) - c(v)| \leq t \Leftrightarrow (u, v) \in E$. Then there is an interval representation of G that contains for each vertex $v \in V$, a unit interval in the range $[c(v)/t, c(v)/t + 1]$. Conversely, if H has a unit interval representation Γ, then scale Γ by an integer factor t, so that each endpoint of each interval has integer coordinate. Then for each vertex v, the coordinate of the left endpoint of the interval for v defines a color $c(v)$ such that for any pair $u, v \in V$, $|c(u) - c(v)| \leq t \Leftrightarrow (u, v) \in E$. The details of this proof is in Appendix A. □

2.3 Graph Sandwich Problem

Problem 5. [9] Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same vertex set V, where $E_2 \subseteq E_1$, and a property Π, does there exist a graph $H = (V, E)$ on the same vertex set such that $E_2 \subseteq E \subseteq E_1$ and H satisfies property Π?

Here E_1 and E_2 can be thought of as *universal* and *mandatory* sets of edges, with E sandwiched between the two sets. We are interested in a particular property for the graph sandwich problem: "proper interval representability". A graph satisfies *proper interval representability* if it admits a proper interval representation.

Lemma 3. *Let $G = (V, E_G)$ and $H = (V, E_H)$ be two graphs on the same vertex set V such that $E_H \subseteq E_G$. Then the threshold-coloring problem for G with respect to the edge partition $\{E_H, E_G - E_H\}$ is equivalent to the graph sandwich problem for the vertex set V, mandatory edge set E_H, universal edge set $E_H \cup (V \times V - E_G)$ and proper interval representability property.*

Proof. (sketch) Define $E_U = E_H \cup (V \times V - E_G)$. Suppose there exists a graph $H^* = (V, E^*)$ such that $E_H \subseteq E^* \subseteq E_U$ and H^* has a proper interval representation. Then by Lemma 2, there exist integers $r \geq 1$ and $t \geq 0$ and a coloring $c : V \to [1 \dots r]$ such that for any pair $u, v \in V$, $|c(u) - c(v)| \leq t$ if and only if $(u, v) \in E^*$. Then for any edge $(u, v) \in E_G$, $(u, v) \in E_H \Leftrightarrow |c(u) - c(v)| \leq t$. Conversely, if there exists integers $r \geq 1$ and $t \geq 0$ such that there is an (r, t)-threshold-coloring c' of G with respect to the edge partition $\{E_H, E_G - E_H\}$, then define an edge set $E^* = \{(u, v) \in V \times V : |c'(u) - c'(v)| \leq t\}$. Clearly the graph $H^* = (V, E^*)$ has an exact (r, t)-threshold-coloring and hence by Lemma 2, H^* has a proper interval representation. Furthermore E^* is sandwiched between E_H and E_U. Details of this proof is in Appendix A. □

The following theorem follows from Lemmas 1, 2 and 3 since both the vertex coloring and the graph sandwich problem for proper-interval-representability are NP-complete [8] and the proper interval recognition can be solved in linear time [4, 13, 14].

Theorem 1. *The* ***Threshold-Coloring*** *and* ***Fixed-Threshold-Coloring Problem*** *are NP-complete while the* ***Exact-Threshold-Coloring Problem*** *can be solved in linear time.*

3 Total-Threshold-Coloring of Graphs

In this section we address the **Total-Threshold-Coloring Problem**: is a given graph G total-threshold-colorable, that is, can every spanning subgraph of G be represented by appropriately coloring the vertices of G?

First note that not every graph (not even every planar graph) is total-threshold-colorable. Suppose that $G = K_4$, and we would like to represent a subgraph where four of the edges remain and span a 4-cycle, while the other two edges are removed (edge-partitioning $\{N, F\}$). Assume that there exists an (r, t)-threshold-coloring with colors c_1, c_2, c_3, c_4 for vertices v_1, v_2, v_3, v_4 respectively. Without loss of generality assume c_4 is the highest color and $(v_1, v_4) \in F$, hence also $(v_2, v_3) \in F$. Also assume $c_3 \geq c_2$ and consequently $c_4 - c_2 \geq c_3 - c_2$. The left side of the inequality should be at

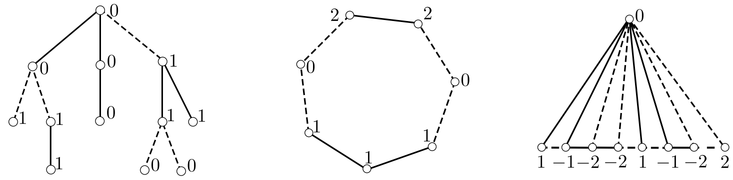


Fig. 2. Threshold-coloring of trees, cycles, and fans.

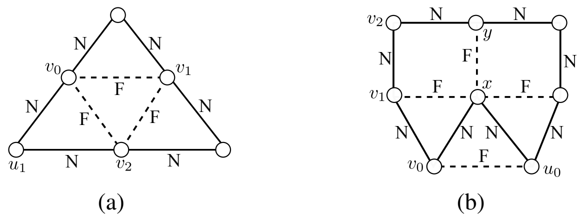


Fig. 3. Graphs which are not threshold-colorable.

most t, and the right side strictly greater than t, which cannot be accomplished by any choice of the range and the threshold.

3.1 Paths, Cycles, Trees, Fans

For *paths* and *trees* there is a trivial coloring with threshold $t = 0$ and two colors. Choose an arbitrary vertex as the root and color it 0. Color 1 all vertices with an odd number of far edges on the shortest path to the root. Color 0 all vertices with an even number of far edges to the root. Then all vertices connected by a near edge of G get the same color, and vertices connected by a far edge get different colors; see Fig. 2(a).

For *cycles* there is a similar coloring scheme with threshold $t = 0$ and three colors. All vertices of a connected component composed of near edges of the cycle get color 2 and the remaining path is colored with 0 and 1 as described above; see Fig. 2(b).

A *fan* is obtained from a path P by adding a new vertex v connected to all vertices of the path. We use threshold $t = 1$ and 5 colors $\{-2, -1, 0, 1, 2\}$ to color a fan. The vertices of P are colored by -1 and 1, and v is colored by 0. After this initial coloring some of the far edges $(u, v), u \in P$ might have $|c(u) - c(v)| = 1$. We fix it by changing the color of u from 1 to 2 or from -1 to -2; see Fig. 2(c).

3.2 Triangular Grid

In a triangular grid all faces are triangles and internal vertices have degree 6. It is easy to show that a triangular grid is not total-threshold-colorable. Consider the graph with vertices $v_0, v_1, v_2, u_0, u_1, u_2$, where each vertex u_i is adjacent to v_{i+1} and v_{i+2} (mod 3); see Fig. 3(a). Let $F = \{(v_0, v_1), (v_1, v_2), (v_2, v0)\}$, and let N contain the remaining 6 edges. Assume that there exists a (r, t)-threshold-coloring c. Without loss of generality, let $c(v_0) < c(v_1) < c(v_2)$. Now on one hand $c(v_2) - c(v_0) > 2t$ and on the other $c(v_2) - c(v_0) \leq |c(v_2) - c(u_1)| + |c(u_1) - c(v_0)| \leq 2t$, which is impossible. This also proves that outerplanar graphs are not total-threshold-colorable in general.

3.3 Hexagonal Grid

In a hexagonal grid all faces are 6-sided and internal vertices have degree 3. Here we show that the hexagonal grid is total-threshold-colorable with $r = 5$ and $t = 1$.

Lemma 4. *Let $P_2 = \{v_0, v_1, v_2\}$ be a path of length 2. Then for any edge-labeling of P_2 and a fixed color $k \in \{-2, -1, 1, 2\}$, there is a threshold-coloring c of P_2 with threshold $t = 1$, where $c(v_0) = 0$, $c(v_2) = k$ and $c(v_1) \in \{-2, -1, 1, 2\}$.*

Proof. Depending on whether the edge (v_0, v_1) is near or far, choose $c(v_1)$ to be 1 or 2. If the label of (v_1, v_2) disagrees with the colors of v_1 and v_2 then we change the sign of $c(v_1)$. □

Lemma 5. *Any hexagonal grid is $(5, 1)$-total-threshold-colorable.*

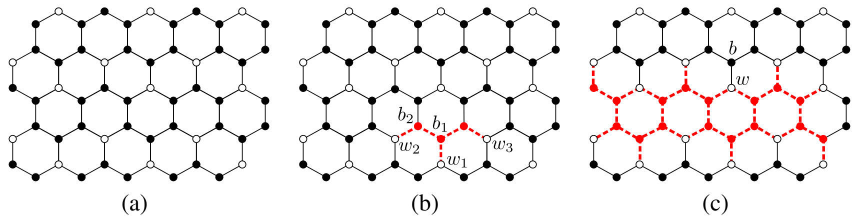


Fig. 4. Total-threshold-coloring of the hexagonal grid. (a) White vertices get color 0, black vertices get one of the colors $-2, -1, 1, 2$. (b) A color assignment to b_1 can be extended to vertices w_2 and w_3 based on the labels of the red dashed edges. (c) The process assigns colors for the red vertices.

Proof. The coloring is done in two steps. In the first step we assign color 0 for a set of independent vertices of G as shown in Fig. 4(a), where the colored vertices are white. Note that no two white vertices have a shortest path of length less than 3.

In the second step we find a coloring of the remaining black vertices, using only four colors $\{-2, -1, 1, 2\}$. Let w_1 be a white vertex. We randomly choose one of its black neighbors b_1, and assign a color for b_1 based on the label of edge (w_1, b_1). Now vertex b_1 has two white vertices w_2 and w_3 within distance 2. Using Lemma 4 we can (uniquely) extend the coloring of b_1 to w_2 (symmetrically, to w_3) so that additional black vertex b_2 gets a color. Again, the coloring of b_2 can be extended to its nearest white neighbor. We continue such a propagation of colors, see Figs. 4(b) and 4(c) where processed black vertices and edges are shown dashed red. One can easily see that the process will color a row of hexagons with alternate upper and lower legs. To complete the coloring of G we choose a white vertex in the next row of hexagons and initiate a similar propagation process. For example, one can use vertices w and b shown in Fig. 4(c). □

3.4 Octagonal-Square Grid

The proof of the following lemma is similar to the proof of Lemma 5, see Appendix B.

Lemma 6. *Any octagonal-square grid is* $(5, 1)$*-total-threshold-colorable.*

3.5 Square-Triangle Grid

We prove that the graph in Fig. 3(b) is not total-threshold-colorable. Assume to the contrary that c is a (r, t)-threshold-coloring. Without loss of generality let $c(v_0) < c(u_0)$. Since (v_1, u_0) is a far edge and $(v_0, x), (u_0, x)$ are near we have $c(v_0) < c(x) < c(u_0)$. Similar argument shows that $c(v_1) < c(v_0) < c(x) < c(u_0) < c(u_1)$. Then if $x < y$, we have $c(v_1) + t < c(x)$ and $c(x) + t < c(y)$, which implies $c(v_1) + 2t < c(y)$. This makes it impossible to find a color for v_2 near to both v_1 and y. Similarly if $x > y$ then it is impossible to color u_2.

Theorem 2 summarizes the results in this section.

Theorem 2. *Paths, cycles, trees, fans, the hexagonal grid and the octagonal-square grid are total-threshold-colorable. The triangular grid and triangle-square grid are not total-threshold-colorable.*

4 Planar Graphs without Short Cycles

In the cases where we have counter-examples of total-threshold-colorability (e.g., K_4 and the triangular grid) we have short cycles, which can be used to force groups of vertices to be simultaneously near and far. In this section we show that if we consider graphs without short cycles, we can prove total-threshold-colorability.

Theorem 3. *Let G be a planar graph without cycles of length ≤ 9. Then G is $(8, 2)$-total-threshold-colorable*[6].

The outline of our proof for Theorem 3 is as follows. We first find some small tree structures T that are "reducible", in the sense that for any edge-labeling of T and any given fixed coloring of the leaves of T to the colors $\{0, 1, \ldots, 7\}$, there is a $(8, 2)$-threshold-coloring of T. For a contradiction assume that there is a planar graph with girth ≥ 10 having no $(8, 2)$-threshold-coloring. We consider the minimal such graph G, and by a discharging argument prove that G contains at least one of these reducible tree structures. This contradicts the minimality of G. We start with some technical claims.

Extending a coloring. Let P_n be a path with vertices $v_0, \ldots, v_n$. Given an edge-labeling of P_n and the color c_0 of v_0 we call a color c_n *legal* if there exists a $(8, 2)$-threshold-coloring c of P_n, so that $c(v_0) = c_0$ and $c(v_n) = c_n$. The proof of the following 3 claims are in Appendix C.

Claim 1 *Let P_1 be a path of length 1. Then at least one of the colors 1 or 6 is legal (irrespective of the edge label and the color c_0).*

Claim 2 *Let P_2 be a path of length 2. Then 3 is legal unless $c_0 = 3$ and $\{N, F\} = \{\{e_1\}, \{e_2\}\}$, i.e. the edges e_1 and e_2 are labeled differently. Symmetrically, 4 is legal unless $c_0 = 4$ and $\{N, F\} = \{\{e_1\}, \{e_2\}\}$.*

Claim 3 *Let P_3 be a path of length 3. Then $1, 3, 4$, and 6 are all legal (irrespective of the edge label and the color c_0).*

A *star* is a subdivision of the graph $K_{1,n}$, and its *center* is the single vertex of degree ≥ 3. Let T be a star. A *prong* of T is a path from a leaf to the center of T, and a prong with k edges is called a k-prong, we say that it has length k.

Claim 4 *Let T be a subdivision of $K_{1,3}$ with prongs of length $1, 2$, and 3, respectively. Assume that the leaves of T are assigned colors, so that the leaf u on the 1-prong is colored with either 1 or 6. Then we can extend this partial coloring to the whole T.*

Proof. Let v be the center of T. Given $c(u)$, we can choose $c(v) \in \{3, 4\}$ so that the labeling condition on the 1-prong is satisfied. If this choice cannot be extended to the longer prongs, then the leaf of the 2-prong is also colored with either 3 or 4, see Claim 2. But then the choice $c(v) \in \{1, 6\}$ which satisfies the labeling condition on the 1-prong can be extended to the remaining prongs. □

Reducible configurations. A *configuration* is a tree T, and is *reducible* if every assignment of colors to the leaves of T can be, for every possible edge-labeling of T, extended to a $(8, 2)$-threshold-coloring c of the whole T.

Claim 5 *A path P_4 of length 4 is a reducible configuration.*

[6] Equivalently, the *girth* (that is, the shortest cycle) of G should be ≥ 10.

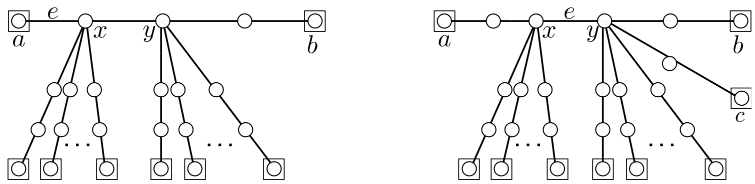


Fig. 5. Two additional types of reducible configurations, T1 and T2.

Proof. Let v be a neighbor of a leaf in P_4. By Claim 1 and Claim 3 either $c(v) = 1$ or $c(v) = 6$ extends to the remaining uncolored vertices. □

Now Claim 5 implies that longer paths are reducible as well. Let us turn our attention to stars.

Claim 6 *(A) Let T be a star with at most 1 prong of length 1 and the remaining prongs have length 3. Then T is reducible.*

(B) Let T be a star with at most 3 prongs of length 2 and the remaining prongs have length 3. Then T is reducible.

Proof. In both cases let v denote the center of the star. In order to establish (A) let $c(v)$ be either 1 or 6, which is appropriate for the 1-prong (such a choice exists by Claim 1). By Claim 3 the coloring $c(v)$ can be extended to the remaining 3-prongs. For (B) we may assume that neither 3 nor 4 can be extended to all three 2-prongs. By Claim 2 both colors 3 and 4 are used at leaves of the 2-prongs. Now, by Claim 1 at least one of $c(v) = 1$ or $c(v) = 6$ extends to the third 2-prong, and hence also to the remaining 2- and 3-prongs, by Claim 2 and Claim 3. □

Claim 7 *There exist two additional types T1 and T2 of reducible configurations shown in Fig. 4.*

The proof of Claim 7 is analogous to the proof of Claim 6, see Appendix C.

Discharging. A *minimal counterexample* is the smallest possible (in terms of order) planar graph G without cycles of length ≤ 9 which is not $(8, 2)$-total-threshold-colorable. A minimal counterexample G does not contain reducible configurations. Further G is connected and has no vertices of degree 1. As G is also not a cycle (such a cycle should be of length ≥ 9 and should not contain a P_4), and is therefore homeomorphic to a (multi)graph of minimal degree ≥ 3.

Let us fix its planar embedding determining its set of faces $F(G)$. Let us define *initial charges*: initial charge of a vertex v, $\gamma_0(v)$, is equal to $4\deg(v) - 10$, and the initial charge of a face f, $\gamma_0(f)$, is equal to $\deg(f) - 10$. A routine application of Euler formula shows that the total initial charge is -20.

As all faces have length ≥ 10, every face is initially non-negatively charged. We shall not alter the charges of faces. The following table shows initial charges of vertices according to their degree:

degree $\deg(v)$	2	3	4	5	6	7	$\cdots$
initial charge $\gamma_0(v)$	-2	2	6	10	14	18	$\cdots$

The discharging procedure will run in two phases, by $\gamma_i(v)$ we shall denote the charge of vertex v after Phase i of discharging. Informally, Phase 1 shall see that vertices of degree 2 do not have negative charges, and Phase 2 will leave only vertices of degree 3 with a possible negative charge.

Let u, v be vertices of G. We say that u and v are 2-adjacent, if G contains a $u - v$-path whose (possible) internal vertices all have degree 2. In Phase 1 we redistribute charge according to the following rule:

Rule 1: every vertex v of degree ≥ 3 sends charge 1 to every vertex u of degree 2, for which v and u are 2-*adjacent*.

In Phase 2 we shall apply the following rule:

Rule 2: If u and v are adjacent with $\gamma_1(u) > 0, \gamma_1(v) < 0$ then u sends charge 1 to v.

As every vertex u of degree 2 (we also call them *2-vertices*) is 2-adjacent to exactly two vertices of bigger degree, we have $\gamma_1(u) = 0$ in this case. For a vertex v of degree ≥ 3, the discharging in Phase 1 decreases the charge of v by the number of 2-vertices which are 2-adjacent to v.

Let v be a vertex of degree ≥ 3. A *prong at* v is a $v-x$-path whose other end-vertex x is of degree ≥ 3 and has internal vertices of degree 2.

Claim 8 *Let v be a vertex of degree ≥ 3. Then the number of 2-vertices that are 2-adjacent to v is at most $2 \cdot \deg(v) - 3$.*

Proof. By Claim 5 each prong at v contains at most two vertices of degree 2. If the shortest prong at v has length 1, then Claim 6 implies that at least one other prong has length ≤ 2. If the shortest prong at v has length 2, then by Claim 6 we have at least four prongs that are of length ≤ 2, and the result follows. □

Now Claim 8 serves as the lower bound for vertex charges after Phase 1, and in turn prepares us for the Phase 2 of discharging.

Claim 9 *(A) Let v be a vertex of degree 3. If $\gamma_1(v) < 0$, then $\gamma_1(v) = -1$ and the prongs at v have lengths $1, 2$ and 3, respectively.*

(B) Let v be a vertex of degree 3. If $\gamma_1(v) = 0$, then the prongs at v have either lengths $1, 1, 3$ or $1, 2, 2$.

(C) Let v be a vertex of degree 3 with its prongs of length $1, 1,$ and 2. Then $\gamma_1(v) = 1$.

(D) Let v be a vertex of degree 3 with all 3 prongs of length 1. Then $\gamma_1(v) = 2$.

(E) If v is a vertex of degree ≥ 4, then $\gamma_2(v) \geq 0$, and also $\gamma_2(v)$ is not smaller than the number of 1-prongs at v.

Proof. Let us first prove (E). Choose a vertex v with $\deg(v) \geq 4$. For every prong of length 3, v sends 2 units of charge in Phase 1. For every shorter prong v sends at most 1 unit of charge in either Phase 1 or Phase 2. The total charge sent out of v in both of the phases is by Claim 6 and Claim 8 at most $2\deg(v) - 2$. Hence $\gamma_2(v) \geq (4\deg(v) - 10) - (2\deg(v) - 2) = 2\deg(v) - 8 \geq 0$.

The other cases merely stratify vertices of degree 3 according to the number of their 2-neighbors of degree 2. □

Now Claim 9(E) states that every vertex v of degree ≥ 4 satisfies $\gamma_2(v) \geq 0$. Similarly, if a 3-vertex u is adjacent to a vertex v whose degree is at least 4, then also $\gamma_2(u) \geq 0$. This fact follows from either Claim 9(A) and (E) (in case $\gamma_1(u) < 0$), or from either Claim 9(C) or (D) (if $\gamma_1(v) > 0$) as in this case u cannot send excessive charge in Phase 2.

Claim 10 *No vertex v has $\gamma_2(v) < 0$ and $\gamma_1(v) < 0$.*

Proof. Let v be a vertex satisfying both $\gamma_2(v) < 0$ and $\gamma_1(v) < 0$. By Claim 9 $\deg(v) = 3$ and v has prongs of length $1, 2, 3$. Let u be the only neighbor of v of degree $\neq 2$. Since

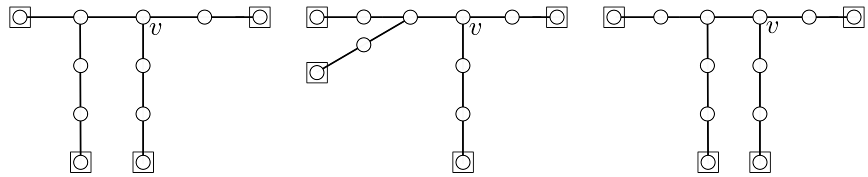


Fig. 6. Negatively charged vertex v after both phases induces a reducible configuration.

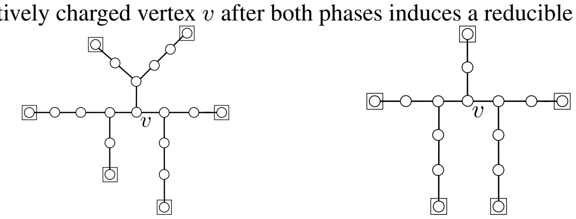


Fig. 7. Negatively charged vertex v after Phase 2, its charge was positive after Phase 1.

v has received no charge from u in Phase 2 we have both $\deg(u) = 3$ and $\gamma_1(u) \leq 0$. By Claim 9 the prongs of u are of lengths $1, 2, 3$ or $1, 1, 3$ or $1, 2, 2$. Hence G contains one of configurations shown in Fig. 6.

Now observe that these are reducible, as each matches one of T1 or T2 types of reducible configurations Claim 7. □

Claim 11 *No vertex v has $\gamma_2(v) < 0$ and $\gamma_1(v) \geq 0$.*

Proof. If $\gamma_1(v) = 0$, then also $\gamma_2(v) = 0$, as Rule 2 does not reduce charge of a discharged vertex. By Claim 9(E) vertices of degree ≥ 4 do not have negative charge after Phase 2.

Hence we may assume that v has degree 3, $\gamma_1(v) > 0$, and $\gamma_2(v) < 0$. By Claim 9(C) and (D) every neighbor u of v satisfies either $\deg(u) = 2$ or $\deg(u) = 3$ and $\gamma_1(u) < 0$. There are exactly two possible cases and they are shown in Fig. 7.

It is enough to see that there exists a color choice $c(v)$ which can be extended in the 2-prong and/or stars centered at neighbors of v.

Let us first settle the option shown in the right. By Claim 1 at least one of $c(v) = 1$ or $c(v) = 6$ extends to the top 2-prong, and this choice also extends to the two copies of T, see Claim 4. The left case is even easier, as both choices $c(v) = 1$ and $c(v) = 6$ extend to the three copies of T, again by Claim 4. □

Now Claim 10 and Claim 11 imply that no vertex has negative charge after Phase 2 of the discharging procedure. As the total charge remains negative and the faces cannot have negative charges we have a contradiction, which completes the proof of Theorem 3.

5 Unit-Cube Contact Representations of Graphs

Lemma 7. *If G has a unit-cube contact representation Γ so that one face of each cube is co-planar in Γ, then any threshold subgraph of G also has a unit-cube representation.*

Proof. Let $H = (V, E_H)$ be a threshold subgraph of $G = (V, E_G)$ and let $c : V \rightarrow [1 \ldots r]$ be an (r, t)-threshold-coloring of G with respect to the edge-partition $\{E_H, E_G - E_H\}$. We now compute a unit-cube contact representation of H from Γ using c.

Assume (after possible rotation and translation) that the bottom face for each cube in Γ is co-planar with the plane $z = 0$; see Fig. 1(a). Also assume (after possible scaling) that each cube in Γ has side length $t + \epsilon$, where $0 < \epsilon < 1$. Then we can obtain a unit-cube contact representation of H from Γ by lifting the cube for each vertex v by an amount $c(v)$ so that its bottom face is at $z - c(v)$; see Fig. 1(b). Note that for any edge $(u, v) \in E_H$, the relative distance between the bottom faces of the cubes for u and v is $|c(u)-c(v)| \leq t < (t+\epsilon)$; thus the two cubes maintain contact. On the other hand, for each pair of vertices u, v with $(u, v) \notin E_H$, one of the following two cases occurs: (i) either $(u, v) \notin E_G$ and their corresponding cubes remain non-adjacent as they were in Γ; or (ii) $(u, v) \in (E_G - E_H)$ and the relative distance between the bottom faces of the two cubes is $|c(u) - c(v)| \geq (t + 1) > (t + \epsilon)$, making them non-adjacent. □

We can directly compute a unit-cube contact representation for subgraphs of the square grid and the hexagonal grid, via geometric algorithms, instead of via threshold coloring [1]. To summarize the results of this section:

Theorem 4. *Any subgraph of the square, hexagonal and octagonal-square grid has a unit-cube contact representation.*

6 Conclusion and Open Problems

We introduced a new graph coloring problem, threshold-coloring, and studied connections with other graph problems. Many interesting open problems remain:

1. Some classes of graphs are total-threshold-colorable, while others are not. There are many classes for which the problem remains open, e.g., the square grid.
2. Planar graphs without cycles of length ≤ 9 are total-threshold-colorable, while all our non-threshold-colorable graphs contain triangles: can this 3-9 gap be reduced?
3. Can we efficiently recognize graphs that are total-threshold-colorable?
4. Is there a good characterization of total-threshold-colorable graphs?
5. The threshold-coloring problem is NP-complete in general. Which restrictions on G and/or H allow it to be polynomial time solvable?

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Appendix

A Proofs of Section 2

Proof (Lemma 2). Suppose a graph $H = (V, E)$ is an exact-threshold graph. This implies that there are integers $r \geq 1, t \geq 0$ and a mapping $c : V \rightarrow [r]$ such that for any pair $u, v \in V$, $|c(u) - c(v)| \leq t \Leftrightarrow (u, v) \in E$. We can find an interval representation of H with unit intervals as follows. Define for each vertex v of H an interval $I(v)$ of unit length where the left-end has x-coordinate $c(v)/t$. Then for any two vertices u and v of H, $I(u)$ and $I(v)$ has a non-empty intersection if and only if $|\frac{c(u)}{t} - \frac{c(v)}{t}| \leq 1 \Leftrightarrow |c(u) - c(v)| \leq t \Leftrightarrow (u, v) \in E$. Thus the intervals gives an interval representation of H.

Conversely, if H has an interval representation Γ with unit intervals, we find an exact (r, t)-threshold-labeling of H for some integers $r \geq 1, t \geq 0$. Scale Γ by a sufficiently large integer factor t such that each endpoint of some interval in Γ has a positive integer x-coordinate (after possible translation in the positive x direction). Let r be the x-coordinate of the right endpoint of the rightmost interval in this scaled representation. Define a labeling $c : V \rightarrow [r]$ where for each vertex v of H, $c(v)$ equals the x-coordinate of the left endpoint of the interval for v. Also define the threshold as the scaling factor t. It is easy to verify that this labeling is indeed an (r, t)-threshold-labeling.

□

Proof (Lemma 3). Let E_U denote the universal edge set $E_H \cup (V \times V - E_G)$ for the graph sandwich problem. Suppose there exists a graph $H^* = (V, E^*)$ such that $E_H \subseteq E^* \subseteq E_U$ and H^* has a proper interval representation. Then by Lemma 2, there exist two integers $r \geq 1$ and $t \geq 0$ and a coloring $c : V \rightarrow [1 \dots r]$ such that for any pair $u, v \in V$, $|c(u) - c(v)| \leq t$ if and only if $(u, v) \in E^*$. We now show that c is in fact a desired threshold-coloring for G. Consider an edge $e = (u, v) \in E_G$. If $e \in E_H$ then $e \in E^*$ since $E_H \subseteq E^*$ and hence $|c(u) - c(v)| \leq t$. On the other hand if $e \in (E_G - E_H)$, $e \notin E_U = E_H \cup (V \times V - E_G)$ and therefore $e \notin E^*$ since $E^* \subseteq E_U$. Hence $|c(u) - c(v)| > t$.

Conversely, if there exists integers $r \geq 1$ and $t \geq 0$ such that there is an (r, t)-threshold-coloring $c : V \rightarrow [1 \dots r]$ of G with respect to the edge partition $\{E_H, E_G - E_H\}$, then define an edge set E^* as follows. For any pair $u, v \in V$, $(u, v) \in E^*$ if and only if $|c(u) - c(v)| \leq t$. Clearly the graph $H^* = (V, E^*)$ has an exact (r, t)-threshold-coloring and hence by Lemma 2, H^* has a proper interval representation. Furthermore for any edge $e = (u, v) \in E_H$, $|c(u) - c(v)| \leq t$ and hence $e \in E^*$. Thus $E_H \subseteq E^*$. Again if $e \in E^*$ then $|c(u) - c(v)| \leq t$. Therefore either $e \in E_H$ or $e \notin E_G \Rightarrow e \in (V \times V - E_G)$. Hence $e \in (E_H \cup (V \times V - E_G)) = E_U$. Thus $E^* \subseteq E_U$. Therefore E^* is sandwiched between the mandatory and the universal set of edges and H^* has a proper interval representation. □

B Proofs of Section 3

Proof (Lemma 6). We use colors $\{-2, -1, 0, 1, 2\}$ and threshold $t = 1$ to find a coloring; the proof is similar to the proof of Lemma 5. We start by partitioning the vertices of the graph into white and black as shown in Fig. 8(a), and we assign color 0 to the white ones. Then we choose a white vertex w and its black neighbor b as in Fig. 8(b),

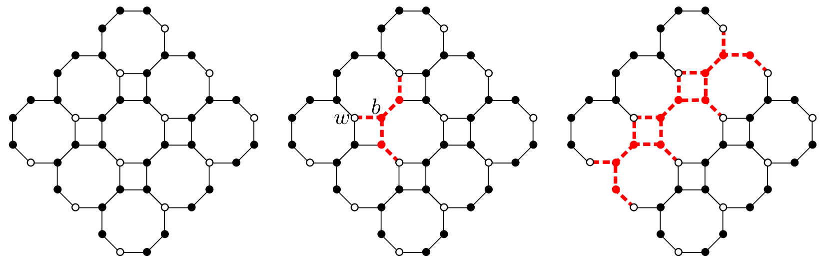

Fig. 8. Total-threshold-coloring of the octagonal-square grid.

and we assign colors $\{-2,-1,1,2\}$ to the "row" of black vertices. It is easy to see that the coloring of rows can be done independently; see Fig. 8(c). □

C Proofs of Section 4

Proof (Claim 1). One only needs to observe that color 1 is close to $0,1,2,3$, and is far from $4,5,6,7$, i.e. the distance between colors is at most 2 or strictly more than 2, respectively. The result follows by symmetry. □

Proof (Claim 2). By symmetry we only give the proof for the case $c_2 = 3$. If $N = \{e_1, e_2\}$ then we choose $c(v_1)$ to be the average of c_0 and c_2, rounding if necessary. If $F = \{e_1, e_2\}$, then one of $0, 7$ is a good choice for $c(v_1)$, as both 0,7 are far from $c_2 = 3$, and at least one is far from c_0. In the remaining case we may assume that $c_0 \neq 3$. If $c_0 < 3$, then set $c(v_1) = 0$ or $c(v_1) = 5$ in case $e_2 \in F$ or $e_2 \in N$, respectively. If $c_0 > 3$, then set $c(v_1) = 6$ or $c(v_1) = 1$ in case $e_2 \in F$ or $e_2 \in N$, respectively. □

Proof (Claim 3). By symmetry it is enough to find appropriate coloring extensions for which $c(v_3) = 1$ and $c(v_3) = 3$. For the latter, choose $c_1 = c(v_1) \neq 3$, according to c_0 and the label of e_1. Now by Claim 2 this choice of c_1 can be extended to the remaining part of P_3, so that $c(v_3) = 3$. The goal $c(v_3) = 1$ splits into two subcases. If $c_0 \neq 3, 4$, then by Claim 2 both 3 and 4 are possible color choices for $c(v_2)$. One is close and the other is far from 1. In case c_0 is either 3 or 4, then again by Claim 2 both 1 and 6 are possible choices for $c(v_2)$. Again, the former is close and the latter is far from 1. □

Proof (Claim 7). Let us first consider the T1 configuration. By Claim 1 one of 1,6 is appropriate for the color of x, with respect to color a and type of edge e. If $b \in \{3, 4\}$, then choose $c(y)$ from $1, 6$, and if $b \notin \{3, 4\}$ then choose $c(y)$ from $\{3, 4\}$. By Claim 2 this works.

Let us now turn to T2. If e is a near edge, we might as well contract e (which implies both x and y will receive the same color), and reduce to Claim 6(B).

Hence we shall assume e is a far edge. Assume first that coloring vertex x with both 1 and 6 extends to the left 2-prong at x. If $c(x) = 1$ and $c(y) = 4$ does not extend to the right 2-prongs at y, we may assume $b = 4$. If $c(x) = 6$ and $c(y) = 3$ does not extend to the right 2-prongs at y, we may assume $c = 3$. In this case setting $c(x) = 1$ and $c(y) = 6$ extends to the right.

By Claim 1 we may assume that only one of $c(x) = 1$ or $c(x) = 6$ extends to the left 2-prong at x, without loss of generality the former. Now $a \neq 3$ and $a \neq 4$, and both $c(x) = 3$ and $c(x) = 4$ extend left. A choice of $c(x) = 1, c(y) = 4$ does not extend to the right 2-prongs at y only if, say, b is equal to 4. But now at least one of $c(y) = 1$ or $c(y) = 6$ extends to the right 2-prongs, and such a choice can be complemented with $c(x) = 4$ or $c(x) = 3$, respectively. □