Tetrahedron Contact Graphs

Md. Jawaherul Alam [*] Muriel Dulieu [†] Justin Iwerks [‡] Joseph O'Rourke [§]

Introduction

Realizing graphs as interior disjoint intersections (contacts) of geometric objects has been a subject of study for many decades. In such a representation the nodes are geometric objects like disks, segments, curves or polytopes in 2D or 3D, while the arcs are realized by two objects touching in some prescribed fashion. Here we study graphs that can be realized as contacts of interior-disjoint tetrahedra. To forbid trivial representations, our contact model requires that no three tetrahedra have a point in common. We call the graphs representable in our model *tetrahedron contact graphs.*

The study of contact graphs dates back to 1936 when Koebe [8] showed that planar graphs can be realized as disk contact graphs. Since then contact graphs in the plane have been extensively studied for curves [6], segments [6], convex shapes [3] and polygons [9]. The *segment contact graphs* [6] and the *triangle contact graphs* [5] are of particular interest since these are also tetrahedron contact graphs. Indeed, using tetrahedra with one face incident to a common plane, one can realize triangle contact graphs as tetrahedron contact graphs. Using "skinny" tetrahedra with one edge incident to a plane, one can realize segment contact graphs as tetrahedron contact graphs.

Contact graphs with 3D objects have been studied for spheres [2, 7], cylinders [1] and axis-aligned rectangular boxes [4]. There is also a rich literature for various 2D and 3D objects S (such as polygons, cylinders, tetrahedra), on finding the number of congruent copies of S that can simultaneously touch S (or each other) [1].

Our Contribution

We study contact graphs with both general and unit-regular tetrahedra. We consider two models of contact: the *unrestricted model* where we allow any type of contact between two tetrahedra; and the *vertex-to-vertex model*, where two tetrahedra touch each other only at a common vertex. Here is a list of our results:

(a) Planar graphs, graphs with ≤ 7 nodes, complete graphs with ≤ 10 nodes, complete bipartite graphs and complete tripartite graphs are contact graphs with general tetrahedra.

(b) Graphs with ≤ 5 nodes and a few graphs with 6 nodes ($K_{2,4}$, $K_{3,3}$) are realizable with tetrahedra in the vertex-to-vertex model.

(c) Complete graphs with ≤ 5 nodes and complete bipartite graphs $K_{m,n}$, $m, n \leq 3$ are contact graphs of unit-regular tetrahedra, but not even all binary trees are realizable.

(d) In the most restricted variant of contacts with unit-regular tetrahedra in the vertex-to-vertex model, K_4 and $K_{2,3}$ are realizable but K_5 and $K_{3,3}$ are not.

We highlight some of these results in the sections below. A forthcoming full paper will contain the proof details.

Irregular Tetrahedra

Here we study contact graphs with no restriction on the shapes and sizes of tetrahedra. We allow for tetrahedra to touch at the vertices, the edges and/or the faces

Theorem 1. *Complete tripartite graphs can be realized as tetrahedron contact graphs.*

Proof. Fig. 1 gives a realization. For the arcs between any two partitions, select an edge from each tetrahedron in both partitions so that the edges from each partition are coplanar. Create adjacencies between these edges by identifying the two planes. □

The corollary below follows from Theorem 1.

Corollary 2. *Complete bipartite graphs can be realized as tetrahedron contact graphs.*

Theorem 3. *Any complete graph K_n, for $n \leq 10$, is a tetrahedron contact graph.*

[*] Dept. of Computer Science, University of Arizona
[†] Dept. of Computer Science and Engineering, Polytechnic Institute of NYU
[‡] Mathematics Dept., The Spence School
[§] Dept. of Computer Science, Smith College

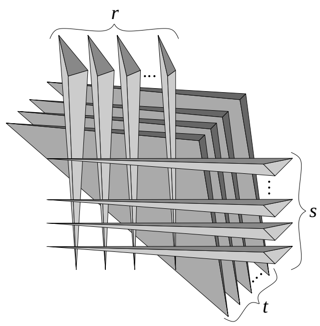

Figure 1: Tetrahedron contact representation for a tripartite graph.

The idea for realizing K_{10} is to make two K_5 graphs and then arrange the tetrahedra so that a complete bipartite graph on these two components is formed.

Theorem 4. *K_5 is realizable in the vertex-to-vertex contact representation of tetrahedra.*

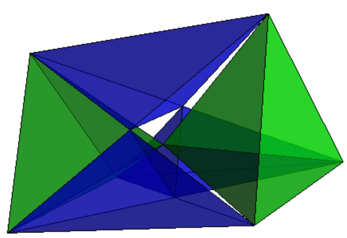

Figure 2: A realization of K_5 in the vertex-to-vertex model with general tetrahedra

Corollary 5. *Graphs of ≤ 5 nodes are realizable in a vertex-to-vertex representation.*

Unit-Regular Tetrahedra

In this section we consider contact representations using regular tetrahedra with unit-length sides.

Lemma 6. *No complete binary tree of height ≥ 17 is realizable with unit tetrahedra.*

Proof. Let T be a complete binary tree with height $h \geq 17$; the number of vertices in T is $n = 2^{h+1} - 1$. Assume that T has a contact representation Γ with unit-regular tetrahedra. Consider the sphere S of the smallest size that bounds Γ. Since the diameter of T is $2h$, the diameter of S is $\leq 2h+1$ and the volume of S is $\leq (2h+1)^3\pi/6$. Since this contains n unit-regular tetrahedra, each with volume $1/6\sqrt{2}$, the volume of S must be $\geq n/6\sqrt{2}$. For $h \geq 17$ this gives a contradiction. □

Theorem 7. *Complete graphs K_n, $n \leq 5$, are realizable with unit-regular tetrahedra.*

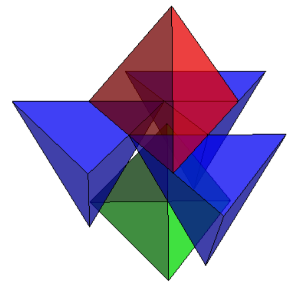

Figure 3: A realization of K_5 with unit-regular tetrahedra.

Acknowledgment

We thank James R. Ashe and Michael Dobbins for useful discussions.

References

[1] A. Bezdek. On the number of mutually touching cylinders. *Combinatorial and Computational Geometry*, 52:121–127, 2005.

[2] K. Bezdek. Sphere packings revisited. *European Journal of Combinatorics*, 27(6):864 – 883, 2006.

[3] J. Czyzowicz, E. Kranakis, D. Krizanc, and J. Urrutia. Discrete realizations of contact and intersection graphs. *Journal of Pure and Applied Mathematics*, 13(4):427–439, 2004.

[4] S. Felsner and M. C. Francis. Contact representations of planar graphs with cubes. In *Symposium on Computational Geometry (SoCG'11)*, pages 315–320, 2011.

[5] E. R. Gansner, Y. Hu, and S. G. Kobourov. On touching triangle graphs. In *Graph drawing (GD'10)*, pages 250–261, 2011.

[6] P. Hliněný. *Contact representations of graphs.* PhD thesis, Charles University in Prague, 2000.

[7] P. Hliněný and J. Kratochvíl. Representing graphs by disks and balls (a survey of recognition-complexity results). *Discrete Mathematics*, 229:101–124, 2001.

[8] P. Koebe. Kontaktprobleme der konformen abbildung. *Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig. Math.-Phys. Klasse*, 88:141–164, 1936.

[9] T. Ueckerdt. *Geometric Representations of Graphs with low Polygonal Complexity.* PhD thesis, Technische Universität Berlin, 2011.