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Fewer Bends Point-set Embedding with Mapping

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***Abstract*—An upward point-set embedding of an upward planar digraph G on a set of points S with a mapping $\Phi : V(G) \rightarrow S$ is an upward planar drawing Γ of G where each vertex of G is placed on a point of S according to Φ. Γ is called an upward topological book embedding of G with the mapping Φ if the points in S are aligned along a straight-line. In this paper, we address the problem of minimizing the total number of bends on the edges in an upward point-set embedding of G with the mapping Φ. We first give an algorithm that finds an upward topological book embedding of G with a mapping if such an embedding exists. Using this result, we then give an algorithm to obtain an upward point-set embedding of G with a mapping if it exists. The drawings obtained by our algorithm for both the problems contain at most (n-3) bends per edge, which improves the previously known upper bound of ($2n$-3) bends per edge. Furthermore we also find an upper bound on total number of bends in our drawing, which is, to the best of our knowledge, the first result on the total number of bends for the point-set embedding problem.**

***Index Terms*—Point-Set Embedding, Topological Book Embedding, Spine Crossing, Minimum Bends, Algorithm.**

I. Introduction

Let G be an upward planar digraph with n vertices and let S be a set of n distinct points in the plane. An *upward point-set embedding* of G on S is an upward planar drawing of G where each vertex of G is placed on a distinct point of S. Suppose Φ is a mapping from the vertices of G to the points of S. An *upward point-set embedding* of G on S with the mapping Φ is an upward point-set embedding of G on S where the vertices of G are located at the points of S according to the mapping Φ [1]. G does not admit an upward point-set embedding on S with every mapping Φ. For example, one can easily observe that there exists no upward point-set embedding of the upward planar digraph G on the points S in Fig. 1 where each vertex v_i of G is mapped to the point i of S for $1 \leq i \leq 4$.

An *upward topological book embedding* of a planar graph G with n vertices is a variant of an upward point-set embedding of G on a set S of n points where the points in S are aligned along a straight-line. Thus an upward book embedding Γ of G takes the form of a 2-page book where the vertices are contained along a straight-line, each edge is drawn in the monotonically increasing direction within the two half-planes induced by the straight-line and the edges are allowed to cross the spine. The straight-line on which the vertices of G are placed is called the *spine* of Γ and the two half-planes induced by the spine are called the *pages* of Γ. Figure 2(b) shows a topological book embedding of the graph G of Fig. 2(a) where the dotted line represents the spine and the two half planes on either side of the spine are the pages of the book. The edge (2, 5) in Fig. 2(b) crosses the spine. Let Φ is an ordering of the vertices of G. An *upward topological book embedding* of G with the ordering Φ is an upward topological book embedding

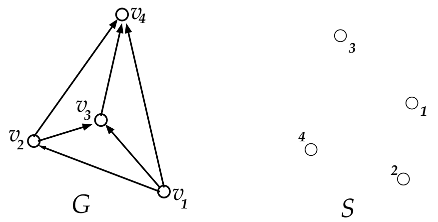

Fig. 1. An upward planar digraph G with four vertices and a set S of four points

of G where the ordering of the vertices along the spine follows Φ.

A *bend* in a drawing of a planar graph G is a point at which an edge of G changes direction. One can easily see that an edge e in an upward topological book embedding Γ of an upward planar digraph can be represented by a polyline with $s + 1$ bends if e crosses the spine s times in Γ. Graph drawing with minimum number of bends is one of the most well studied research topics for the current researchers. Kaufman and Wiese developed two algorithms to draw a four-connected plane graph with at most one bend per edge and a general plane graph with at most two bends per edge [2]. For the class of outerplanar triangulated st-digraphs, a recent result gives a linear-time algorithm for an upward topological book embedding with the minimum number of bends where at most two bends per edge are allowed [4]. However neither of these algorithms has maintain a given point-set embedding of the input graph. On the other hand, the problem of computing point set embeddings of planar graphs with small number of bends also has a rich collection of literature [2], [5], [6]. Giordano *et al.* proved that any planar graph has an upward topological book embedding Γ such that every edge of G contains at most two bends in Γ and with the help of this result, they have developed an algorithm to obtain an upward point-set embedding of an upward planar digraph on any set of n distinct points in the plane but their algorithm does not minimize the total number of bends in Γ [3]. Furthermore, none of the algorithms mentioned so far has considered a given mapping of the vertices to the points. Then Giordano, Liotta and Whitesides modify the previous algorithm to give an algorithm to draw an upward point-set embedding of an upward planar digraph with a given mapping with at most ($2n$-3) bends per edge [1]. They posted an open problem of minimizing the total number of bends of an upward point-set embedding in the paper and we address the problem here. In this paper we develop an algorithm to draw an upward topological book embedding of an upward planar digraph according to a given mapping with at most (n-3) bends per edge if such an embedding exists. We show that

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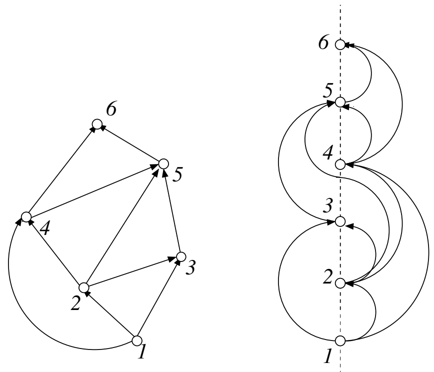

(a) (b)

Fig. 2. (a) A planar graph G, (b) a topological book embedding of G with a spine crossing by the edge $(2, 5)$.

$2(n-4) + 3(n-5) + \ldots + k(n-2-k) + p(n-3-k)$ spine crossings are sufficient for our drawing where k and p are integers, the total number of edges crossing the spine is $k(k+1)/2 + p$ and $p < k$. Using this result, one can easily find an upper bound on the minimum total number of bends in an upward point-set embedding of an upward planar digraph with a given mapping. The rest of the paper is organized as follows. In Section 2, we define some basic terminologies used throughout the paper. Section 3 describes our algorithm for upward topological book embedding of an upward planar digraph with a mapping. Section 4 gives our result on upward point-set embedding with a given mapping using the ideas of the algorithm on upward topological book embedding. Finally conclusions and possible directions for future research are covered in Section 5.

II. Preliminaries

In this section we present some definitions and terminologies.

Let G=(V, E) be a simple graph with the vertex set V and the edge set E. A digraph is a directed graph where each edge has a direction. We denote a directed edge joining the vertices u and v of G by (u, v). If $(u, v) \in E$, then the vertex v is said to be *adjacent* to u and the edge (u, v) is said to be incident to u and v. A graph is *planar* if it can be embedded in a plane so that no two edges intersect each other geometrically except at a vertex to which they are both incident. A plane graph is a planar graph with a fixed planar embedding.

A topological ordering of a planar digraph G with n vertices is a mapping ρ of its vertices to distinct integers such that for every edge (u, v), we have $\rho(u) < \rho(v)$. A topological numbering is a topological ordering where the vertices are mapped to integers $1, 2, \ldots, n$. A topological numbering of a planar digraph with n vertices can be computed in $O(n)$ time using standard graph search techniques [7].

A *book* is defined as a collection of half planes called pages which join together at a straight-line called the *spine*. An *upward topological book embedding* of a planar graph G is a drawing of G on a book with two pages such that the vertices are aligned along the spine, the edges are drawn as upward on the pages and are allowed to cross the spine. For the rest of the paper, we shall assume that in a topological book embedding of a planar graph, the spine is aligned along the y-axis. In such a drawing the half plane on the left hand side of the spine is called the *left page* and that on the right hand side is called the *right page*.

III. Upward Topological Book Embeddings

In this section, we describe our algorithm to obtain an upward topological book embedding Γ of an upward planar digraph G with a given ordering Φ of the vertices along the spine.

It is trivial to see that G admits an upward topological book embedding with the ordering Φ if and only if Φ induces a topological numbering of $V(G)$. We thus assume that Φ induces a topological numbering of $V(G)$. We first rename the vertices so that the vertices are labeled as $v_1, v_2, \ldots, v_n$ in the order of Φ. We now have the following lemma.

Lemma 3.1: Let G be an upward planar digraph of n vertices with a directed hamiltonian path P and let Φ be a topological ordering of the vertices of G. Then one can find an upward topological book embedding of G in linear time with no spine crossing.

Proof: Let us rename the vertices so that the vertices are labeled as $v_1, v_2, \ldots, v_n$ in the order of Φ. Then P contains only the edges (v_i, v_{i+1}) for $1 \leq i < n$ since Φ is a topological ordering of G. Let G' be a plane embedding of G. We now obtain a topological book embedding Γ of G with no spine crossing as follows. We first fix the position of the vertices on a vertical straight-line L (the spine of Γ) such that v_{i+1} is above v_i for $1 \leq i < n$. We first draw each edge of P by a semi-circle between its two end-vertices on either the left page or the right page of Γ. We draw the rests of the edges as follows. If an edge is to the left (right) of P in G', then we draw it on the left (right) page of Γ by a semicircle between its end-vertices. Clearly Γ keeps the embedding G' unaltered and hence Γ is a planar drawing of G. Furthermore it is obvious from the algorithm that all the vertices of G are on the straight-line L and each edge is either on the left page or on the right page of Γ, creating no spine crossing. Thus Γ is an upward topological book embedding of G with no spine crossing. It is also trivial to implement the algorithm in linear time. $\mathcal{Q.E.D.}$

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Let G be an upward planar digraph and let Φ be a topological ordering of the vertices of G. If G contains a hamiltonian path, then the proof of Lemma 3.1 gives a linear-time algorithm to obtain an upward topological book embedding of G with the ordering according to Φ. We call this algorithm **Draw_Ham** for the rest of this section. We now use this algorithm to obtain a topological book embedding of any upward planar digraph G with the ordering Φ. We have the following theorem.

Theorem 3.1: Let G be an upward planar digraph with n vertices and let Φ be a topological ordering of $V(G)$. Then one can find an upward topological book embedding of G with the ordering Φ in $O(n^2)$ time where each edge of G crosses the spine at most (n-4) times. Furthermore, if s is the number of edges that crosses the spine, then the total number of spine crossings is at most $2(n-4)+3(n-5)+\ldots+k(n-2-k)+p(n-3-k)$ where k and p are integers, $s = k(k+1)/2-1+p$ and $p < k$.

In the rest of this section, we give a constructive proof of Theorem 3.1.

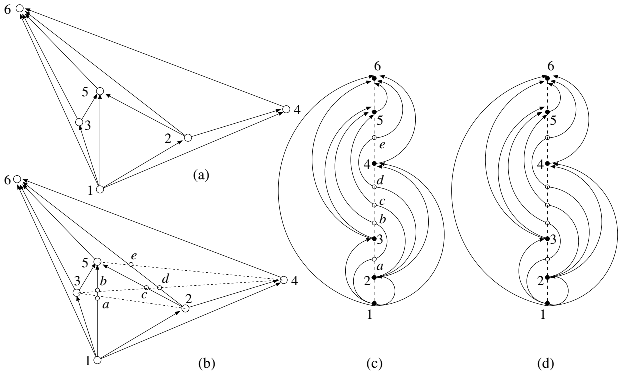

Fig. 3. Illustration of Algorithm **Draw_General**.

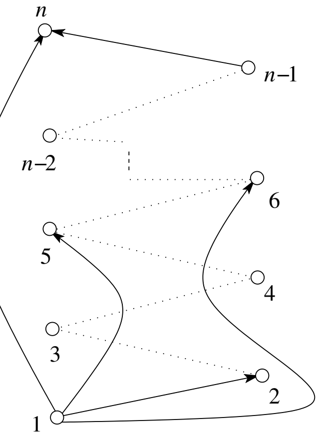

Fig. 4. Illustration of the proof of Lemma 3.2

We rename the vertices of G so that the vertices are labeled as $v_1, v_2, \ldots, v_n$ in the order of Φ. Let λ be an upward planar stright-line drawing of G as illustrated in Fig. 3(a). For every vertex v of G, we denote by $p(v)$ the point on which v is placed in λ. We may assume that for any three vertices u, v and w, $p(u)$, $p(v)$ and $p(w)$ are not collinear. We now construct an upward planar digraph G' with a hamiltonian path from G as follows. If G contains the edge (v_i, v_{i+1}) for $1 \leq i < n$, then G itself has a hamiltonian path. We thus assume that there is an index i ($1 \leq i < n$) such that G does not contain the edge (v_i, v_{i+1}). If the straight-line segment between $p(v_i)$ and $p(v_{i+1})$ does not intersect any existing edge in λ, then we add the edge (v_i, v_{i+1}) to G. Otherwise, suppose that the straight-line segment between $p(v_i)$ and $p(v_{i+1})$ intersects the edges $e_1 = (u_1, w_1)$, $e_2 = (u_2, w_2)$, $\ldots$, $e_f = (u_f, w_f)$ in this order from $p(v_i)$ to $p(v_{i+1})$. Then we subdivide all these edges, i.e. we delete the edges e_1, e_2, $\ldots$, e_f from G; add the dummy vertices x_1, x_2, $\ldots$, x_f to G; and add the edges (u_1, x_1), (x_1, w_1), (u_2, x_2), (x_2, w_2), $\ldots$, (u_f, x_f), (x_f, w_f). Finally we add the edges (v_i, x_1), (x_1, x_2), $\ldots$, (x_{f-1}, x_f) and (x_f, v_{i+1}). For each index i ($1 \leq i < n$) such that G does not contain the edge (v_i, v_{i+1}), we do the same operations as above. If p is the total number of dummy vertices, then the resulting graph G' has $n+p$ vertices and contains a hamiltonian path as illustrated in Fig. 3(b). Clearly $p = O(n^2)$. Then Algorithm **Draw_Ham** gives an upward topological book embedding Γ' of G' with no spine crossing in $O(n^2)$ time as illustrated in Fig. 3(c). We now obtain a topological book embedding Γ of G from Γ' by deleting the dummy vertices as well as the edges incident to them and drawing the edges of G that were deleted in G' through the dummy vertices as illustrated in Fig. 3(d). Note that the dummy vertices of G' represents the spine crossings in Γ. For the rest of this article, we call this algorithm **Draw_General**. We now have the following two lemmas.

Lemma 3.2: Let G be an upward planar digraph and let $\Phi = v_1, v_2, \ldots, v_n$ be a topological ordering of G. Let Γ be the upward topological book embedding of G with the ordering Φ obtained by Algorithm **Draw_General**. Then the edge (v_i, v_j) can cross the spine at most (j-i-2) times.

Proof: Let λ be any upward planar straight-line drawing of G and let $p(v)$ denote the point on which a vertex v of G is placed in Γ. Let L denote the polyline containing the straight-line segments between $p(v_i)$ and $p(v_{i+1})$ for $1 \leq i < n$ as illustrated in Fig. 4. Since the drawing is upward, the edge (v_i, v_j) can cross at most (j-i) line-segments of L between $p(v_i)$ and $p(v_j)$ in Γ. However since the edge is drawn as a straight-line segment, it does not cross the two line-segments between $(p(v_i), p(v_{i+1}))$ and between $(p(v_{j-1}), p(v_j))$. Thus from the algortihm it is obvious that the edge (v_i, v_j) can cross the spine at most (j-i-2) times. $\mathcal{Q.E.D.}$

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Lemma 3.3: Let G be an upward planar digraph and let Φ be a topological ordering of G. Let Γ be the upward topological book embedding of G with the ordering Φ obtained by Algorithm **Draw_General**. Then there are at most $k(k+1)/2 - 1$ edges each of which can cross the spine at least (n-2-k) times for $k \geq 1$.

Proof: Let (v_i, v_j) be an edge that crosses the spine at least (n-2-k) times. Then by Lemma 3.2, $j - i \geq n - 2 - k + 2 = n - k$. Thus there are at most k edges ((v_1, v_{n-k+1}), (v_1, v_{n-k+2}), $\ldots$, (v_1, v_n)) from v_1 that crosses the spine at least (n-2-k) times in Γ. Similarly there are at most $k-1$ edges from v_2 that crosses the spine at least (n-k-2) times in Γ and so on. Thus the number of edges that crosses the spine at least (n-k-2) times in Γ is at most $k + (k-1) + \ldots + 1 = k(k+1)/2$. However if the edge (v_1, v_n) is contained in G, we may assume that it is on the outer face in Γ since otherwise we can redraw it keeping the edge (v_1, v_n) on the outerface. Thus the edge (v_1, v_n) does not cross the spine and the result follows. $\mathcal{Q.E.D.}$

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We are now ready to prove Theorem 3.1

Proof of Theorem 3.1. Let Γ be the upward topological book embedding of G with the ordering Φ obatined by Algorithm **Draw_General**. By Lemma 3.3, there is no edge that crosses the spine at least (n-3) times in Γ. Thus an edge can cross the spine at most (n-4) times in Γ. Furthermore, if the total number of edges that crosses the spine in Γ is at most s, then by Lemma 3.3, the total number of spine crossings is at most $2(n-4)+3(n-5)+\ldots+k(n-2-k)+p(n-3-k)$ where k and p are integers, $s = k(k+1)/2 - 1 + p$ and $p < k$. Finally since the number of spine crossings is bounded by $O(n^2)$, the time complexity of the algorithm is also $O(n^2)$. $\mathcal{Q.E.D.}$

Let G be an upward planar digraph and let Φ be a topological ordering of the vertices of G. Let Γ be the upward topological book embedding of G with the ordering Φ obtained by Algorithm **Draw_General**. Since (v_1, v_2), (v_{n-1}, v_n) and

(v_1, v_n) does not cross the spine in Γ, at most ($3n$-9) edges crosses the spine in Γ. Theorem 3.1 then gives an upper bound on the total number of spine crossings in Γ. However in reality, the number of edges that crosses the spine is much less than the trivial bound of ($3n$-9). We finish this section with the following conjecture the proof of which can give a much better upper bound on the total number of spine crossings.

Conjecture 3.1: Let G be an upward planar digraph and let Φ be a topological ordering of G. Let Γ be the upward topological book embedding of G with the ordering Φ obtained by Algorithm **Draw_General**. Then there are at most ($2n$-6) edges of G that crosses the spine in Γ.

IV. Upward Point-Set Embedding

In this section, we address the problem of finding an upward point-set embedding of an upward planar digraph G of n vertices on a set of n distinct points on the plane with a mapping from the vertices of G to the points in S. Giordano *et. al.* gave an approach to study this problem using algorithms for upward topological book embeddings of G with a given ordering of the vertices of G [1]. We extend their results using Algorithm **Draw_General**. We have the following theorem.

Theorem 4.1: Let G be an upward planar digraph with n vertices and let S be a set of n distinct points in the plane. Let Φ be a mapping from the vertices of G to the points in S. Then G admits an upward point-set embedding with the mapping Φ if and only if there exists a directed line l such that Φ induces a topological ordering of G on l. Furthermore, such an upward point-set embedding of G on S can be computed in $O(n^2)$ time with at most (n-3) bends per edge.

Proof: The proof follows the same line of reasoning as in [1]. The only difference is that we use Algorithm **Draw_General** for finding an upward topological book embedding Γ of G. The upward point-set embedding obtained by algorithm in [1] uses at most one more bend for each edge than the number of spine crossings it induces in Γ. Thus an edge in Γ can contain at most (n-3) bends. $\mathcal{Q.E.D.}$

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One can also find an upper bound on the total number of bends in an upward point-set embedding of G from the fact that for each edge, the number of bends in the point-set embedding is at most one more than the number of spine crossings for that edge in Γ and that there are at most ($3n$-6) edges in G. Furthermore, Giordano *et. al.* also gave an $O(n^3)$-time testing algorithm to check whether an upward planar digraph G of n vertices admits an upward point-set embedding on a set S of n distinct points with a given mapping from the vertices of G to the points in S.

V. Conclusion

In this paper we gave a quadratic-time algorithm to find an upward point-set embedding of an upward planar digraph G of n vertices on a set S of n points with a mapping from the vertices of G to the points in S. We gave an upper bound on the number of bends per edge and the total number of bends for the upward point-set embedding of G, which improves the previously known upper bounds. Recently, Mchedlidze and Symvonis have developed an algorithm for *"ρ-constrained upward topological book embedding"* of an embedded planar st-digraphs which can also be used to find the same bound on the number of bends per edge for upward point-set embedding of directed planar graphs [8]. Minimizing the total number of bends in an upward point-set embedding for a given upward planar digraph with a given mapping remains as an open problem.

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