

Threshold-Coloring and Unit-Cube Contact Representation of Planar Graphs

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Abstract

In this paper we study *threshold-coloring* of graphs, where the vertex colors represented by integers are used to describe any spanning subgraph of the given graph as follows. A pair of vertices with a small difference in their colors implies that the edge between them is present, while a pair of vertices with a big color difference implies that the edge is absent. Not all planar graphs are threshold-colorable, but several subclasses, such as trees, some planar grids, and planar graphs with no short cycles can always be threshold-colored. Using these results we obtain unit-cube contact representation of several subclasses of planar graphs. Variants of the threshold-coloring problem are related to well-known graph coloring and other graph-theoretic problems. Using these relations we show the NP-completeness for two of these variants, and describe a polynomial-time algorithm for another.

Keywords: graph coloring, threshold-coloring, planar graphs, unit-cube contact representation

2010 MSC: 05C15, 05C78

1. Introduction

Graph coloring is among the fundamental problems in graph theory. Typical applications of the problem and its generalizations are in job scheduling, channel assignments in wireless networks, register allocation in compiler optimization and many others [1]. In this paper¹ we consider a new graph coloring problem in which we assign colors (integers) to the vertices of a graph G in order to define a spanning subgraph H of G . In particular, we color the vertices of G so that for each edge of H , the two endpoints are near, that is, their difference is at most a given “threshold”, and for each edge

¹A part of the results of this paper was presented at the *39th International Workshop on Graph-Theoretic Concepts in Computer Science (WG'13)* [2] and *7th International Conference on Fun With Algorithms (FUN'14)* [3].

of $G \setminus H$, the endpoints are far, that is, their difference is greater than the threshold; see Figure 1.

The motivation of the problem is severalfold. First, such coloring arises in the context of the geometric problem of *unit-cube contact representation* of planar graphs. In such a representation of a graph, each vertex is represented by a unit-size cube and each edge is realized by a common boundary with non-zero area between the two corresponding cubes. Finding classes of planar graphs with unit-cube contact representation was recently posed as an open question by Bremner et al. [4]. In this paper we partially address this problem as an application of our coloring problem in the following way. Suppose a planar graph G has a unit-cube contact representation where one face of each cube is co-planar; see Figure 1(a). Assume that we can define a spanning subgraph H of G by our particular vertex coloring. We show that it is possible to compute a unit-cube contact representation of H by lifting the cube for each vertex v by the amount equal to the color of v (where the size or side-length of the cubes are roughly equal to the threshold); see Figure 1(b).

Another motivation for the threshold-coloring comes from the notion of *adjacency labeling scheme* [5, 6]. The idea is to label (color) vertices of a graph in a way that will allow one to infer the adjacency of two vertices directly from their labels without using additional information. Clearly labels of unrestricted size can be used to encode any desired information. However, for practical considerations, it is important to keep labels relatively short and allow efficient information deduction. Threshold-coloring makes it possible to determine the adjacency between two vertices of a graph in constant time. Finally, such coloring can be used for the Frequency Assignment Problem [7], which asks for assigning frequencies to transmitters in radio networks so that only specified pairs of transmitters can communicate with each other.

1.1. Problem Definition

An *edge-labeling* of graph $G = (V, E)$ is a mapping $\ell : E \rightarrow \{N, F\}$ assigning labels N or F to each edge of the graph; we informally name edges labeled with N as the *near* edges, and edges labeled with F as the *far* edges. Note that such an edge-labeling of G defines a partition of the edges E into near and far edges. By abuse of notation the pair $\{N, F\}$ also denotes this partition.

Let $r \geq 1$ and $t \geq 0$ be two integers and let $[1 \dots r]$ denote a set of r consecutive integers. For a graph $G = (V, E)$ and an edge-labeling $\ell : E \rightarrow \{N, F\}$ of G , an (r, t) -*threshold-coloring* of G with respect to ℓ is a *coloring* $c : V \rightarrow [1 \dots r]$ such that for each edge $e = (u, v) \in E$, if $(u, v) \in N$ then $|c(u) - c(v)| \leq t$ and if $(u, v) \in F$ then $|c(u) - c(v)| > t$. We call r and t the *range* and the *threshold*, respectively. Note that the set of near edges defines a spanning subgraph $H = (V, N)$ of G , where H is a *spanning subgraph* of graph G if it contains all vertices of G . We can thus redefine threshold-coloring for a pair (G, H) of a graph $G = (V, E_G)$ and a spanning subgraph $H = (V, E_H)$ of G : an (r, t) -*threshold-coloring* for (G, H) is the one for G with respect to the labeling $\ell : E_G \rightarrow \{N, F\}$, where $\ell(e) = N$ if $e \in H$ and $\ell(e) = F$ if $e \notin H$. The graph H is a *threshold-subgraph* of G if there exists an (r, t) -*threshold-coloring* for (G, H) for some integers r, t .

A graph G is (r, t) -*total-threshold-colorable* for some $r \geq 1, t \geq 0$, if for every edge-labeling ℓ of G there exists an (r, t) -*threshold-coloring* of G with respect to ℓ .

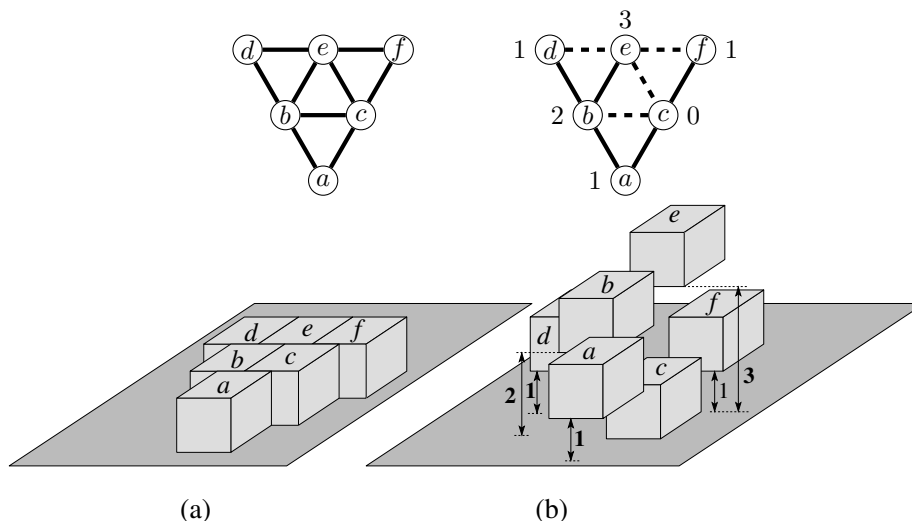


Figure 1: (a) A planar graph $G = (V, E_G)$ and a corresponding unit-cube contact representation where the bottom faces of all cubes are co-planar, (b) a spanning subgraph $H = (V, E_H)$ of G with a $(4, 1)$ -threshold-coloring for (G, H) and a corresponding unit-cube contact representation. The edges in $E_G \setminus E_H$ (far edges) are shown dashed, and the edges in E_H (near edges) are shown solid.

Informally speaking, for every partition of edges of G into near and far edges, we can produce vertex colors so that endpoints of near edges receive near colors, and endpoints of far edges receive colors that are far apart. A graph G is *total-threshold-colorable* if it is (r, t) -total-threshold-colorable for some range $r \geq 1$ and threshold $t \geq 0$. In this paper we focus on the following problem variants.

Total-Threshold-Coloring: Given a graph G , is G total-threshold-colorable, that is, is every spanning subgraph of G a threshold subgraph of G ?

The problem is closely related to the question about whether a particular spanning graph H of G is threshold-colorable.

Threshold-Coloring: Given a graph G and a spanning subgraph H , is H a threshold subgraph of G for some integers $r \geq 1, t \geq 0$?

Another interesting variant of the threshold-coloring is the one in which we specify that the graph G is the complete graph. In this case we call H an *exact-threshold* graph if H is a threshold subgraph of the complete graph G for some integers $r \geq 1, t \geq 0$.

Exact-Threshold-Coloring: Given a graph H , is H an exact-threshold graph?

In the final variant of the problem we assume that the threshold and the range are the part of the input.

Fixed-Threshold-Coloring: Given a graph G , a spanning subgraph H , and integers $r \geq 1, t \geq 0$, is H (r, t) -threshold-colorable?

1.2. Related Work

Many problems in graph theory deal with coloring or assigning labels to the vertices of a graph [1]; many graph classes are defined based on such coloring and labeling; see [8] for an excellent survey. To the best of our knowledge, total-threshold-colorability defines a new class of graphs. Here we mention two closely related classes: threshold graphs and difference graphs. *Threshold graphs* are ones for which there is a real number S and for every vertex v there is a real weight a_v such that (v, w) is an edge if and only if $a_v + a_w \geq S$ [9]. A graph is a *difference graph* if there is a real number S and for every vertex v there is a real weight a_v such that $|a_v| < S$ and (v, w) is an edge if and only if $|a_v - a_w| \geq S$ [10]. Note that for both classes the threshold (real number S) defines edges between all pairs of vertices, while in our setting the threshold defines only the edges of a graph G , which is not necessarily a complete graph. Both threshold and difference graphs can be characterized in terms of forbidden induced subgraphs. For our problem such a characterization is unknown. For details on threshold and difference graphs, see [9].

Threshold-colorability is related to the *integer distance graph* representation [11, 12]. An integer distance graph is a graph with the set of integers as vertex set and with an edge joining two vertices u and v if and only if $|u - v| \in D$, where D is a subset of the positive integers. Clearly, an integer distance graph is an exact-threshold graph if the set D is a set of consecutive integers. Another related graph coloring problem is the *distance constrained graph labeling*. Here the goal is to find a so-called $L(p_1, \dots, p_k)$ -labeling of the vertices of a graph which means that for every pair of vertices at distance at most $i \leq k$ we have that the difference of their labels is at least p_i . The most studied variant is an $L(2, 1)$ -labeling [13, 14]. Minimizing the number of labels in $L(2, 1)$ -labeling is NP-complete, even for graphs with diameter 2 [13]. It is also NP-complete to determine whether an $L(2, 1)$ -labeling exists with at most k labels for every fixed integer $k \geq 4$ [15]. Exact exponential algorithms for $L(2, 1)$ -labeling are described in [16].

A threshold-coloring of a planar graph can be used to find a contact representation of the graph with cuboids (axis aligned boxes) in 3D. Thomassen [17] shows that any planar graph has a proper contact representation by cuboids in 3D. In a *contact representation* of a graph, the vertices are represented by interior-disjoint cuboids (or other polygonal shapes) and the edges are realized by a common boundary of the two corresponding cuboids. A contact representation is *proper* if for each edge the corresponding common boundary has non-zero area. Felsner and Francis [18] prove that any planar graph has a (non-proper) contact representation by cubes. Bremner et al. [4] proves that the same result does not hold when using only unit cubes. Our results on threshold-coloring of planar graphs translates to results on classes of planar graphs that can be represented by contact of unit cubes.

1.3. Our Contribution

First we study the relation of the various threshold-coloring problems with other graph-theoretic problems. Specifically, we show that **Threshold-Coloring** and **Exact-Threshold-Coloring** are NP-complete by reductions from a graph sandwich problem







| Graph Class | trees, cycles*, fans | square grid | hexagonal, octagonal-square grids | triangle-dodec., square-hexagon-dodec. grids | triangular, square-triangle Grids | girth-10 planar graphs |
|--------------------|---|---|---|---|---|---|
| |  |  |  |  |  |  |
| threshold coloring | $r = 5,$ $t = 1$ | Open, $r = O(n)$ | $r = 5,$ $t = 1$ | $r = 9,$ $t = 2$ | No | $r = 8,$ $t = 2$ |
| unit-cube contact | No [4]* | Yes [2] | Yes | Yes | Open | No [4] |

Table 1: Results on **Total-Threshold-Coloring** problem. “No” entries in the last row follow [4] from the fact that graphs with high-degree (e.g., > 14) vertices have no unit-cube representation. * Cycles have unit-cube contact representations.

and the classical vertex coloring problem, respectively. We also show that **Fixed-Threshold-Coloring** can be solved in linear time since it is equivalent to the proper interval graph recognition problem.

We then investigate the **Total-Threshold-Coloring** problem for various subclasses of graphs. In particular, we show that trees, cycles, fans, and planar graphs with girth (length of shortest cycle) at least 10 are always total-threshold-colorable. On the other hand, we provide several examples of planar graphs having an edge-labeling admitting no threshold-coloring. Then we study the problem for regular planar grids. We prove that some of them are total-threshold-colorable (e.g., hexagonal, octagonal-square), while the triangular and square-triangle grids are not. Finally, for the square grid, no constant range of colors suffices. Our results are summarized in Table 1.

As an application of the threshold-coloring problem, we address the problem of contact representation of planar graphs with unit cubes. Given a planar graph, we investigate whether each of its subgraphs has such a representation. We show how we can use the threshold-coloring for computing unit-cube contact representations for some subclasses of planar graphs. Thus we answer some of the open problems from [4] for the subclasses of planar graphs. The last row of Table 1 summarizes these results.

2. Threshold-Coloring and Other Graph Problems

We begin by showing the connections between threshold-colorability and some classical graph-theoretical and graph coloring problems.

2.1. Vertex Coloring Problem

Let $G = (V, E)$ be a graph. We call G k -vertex-colorable if there exists a coloring $c : V \rightarrow [1 \dots k]$ such that for any edge $(u, v) \in E$, $c(u) \neq c(v)$, that is, u and v have different colors. Given an input graph G and an integer $k > 0$, the vertex coloring problem asks whether there exists a k -vertex-coloring of G .

Observation 1. *Let $G = (V, E)$ be a graph and let k be a positive integer. Define an edge-labeling $\ell : E \rightarrow \{N, F\}$ that assigns each edge the label F , that is, for each edge $e \in E$, $\ell(e) = F$. Then G has a k -vertex-coloring if and only if there exists a $(k, 0)$ -threshold-coloring of G with respect to ℓ .*

PROOF. Let $c : V \rightarrow [1 \dots k]$ define a mapping of the vertices of G to the colors $[1 \dots k]$. Then c is a k -vertex-coloring of G if and only if for each edge $e = (u, v) \in E$, $c(u) \neq c(v)$. This is equivalent to saying $|c(u) - c(v)| > 0$, or in other words c is a $(k, 0)$ -threshold-coloring of G with respect to ℓ .

2.2. Proper Interval Representation Problem

An *interval representation* [8] for a graph $G = (V, E)$ is one where each vertex v of G is represented by an interval $I(v)$ of \mathbb{R} such that for any pair $u, v \in V$, the intervals $I(u)$ and $I(v)$ have a non-empty intersection, that is, $I(u) \cap I(v) \neq \emptyset$ if $(u, v) \in E$; otherwise $I(u)$ and $I(v)$ are disjoint, that is, $I(u) \cap I(v) = \emptyset$. A *proper interval representation* [8] for G is an interval representation of G where no interval properly contains another. A *proper interval graph* is one that has a proper interval representation. Equivalently, a *proper interval graph* is one that has an interval representation with unit intervals [19]. The problem of proper interval representation for a graph G asks whether G has a proper interval representation. The problem has been studied extensively [19–21], and it still attracts attention [22].

Lemma 1. *A graph is an exact-threshold graph if and only if it is a proper interval graph.*

PROOF. Let graph $H = (V, E)$ be an exact-threshold graph. This implies that there are integers $r \geq 1, t \geq 0$ and a mapping $c : V \rightarrow [1 \dots r]$ such that for any pair $u, v \in V$, $(u, v) \in E \Leftrightarrow |c(u) - c(v)| \leq t$. We can rephrase this as the equivalent statement $|c(u) - c(v)| < t + \epsilon$ for some $0 < \epsilon < 1$. We can find an interval representation of H with unit intervals as follows. Choose an arbitrary ϵ such that $0 < \epsilon < 1$. Define for each vertex v of H an interval $I(v)$ of unit length where the left-end has x -coordinate $c(v)/(t + \epsilon)$. Then for any two vertices u and v of H , $I(u)$ and $I(v)$ has a non-empty intersection if and only if $|\frac{c(u)}{t+\epsilon} - \frac{c(v)}{t+\epsilon}| \leq 1$, which is equivalent to saying that $|c(u) - c(v)| \leq t + \epsilon$. But $c(u), c(v)$ are integers so we have $|c(u) - c(v)| \leq t$. Then $I(u)$ and $I(v)$ has non-empty intersection if and only if (u, v) is an edge of H . Thus these intervals yield an interval representation of H .

Conversely if H is a proper interval graph, then there is an interval representation Γ for H with unit intervals such that the endpoints of each interval in Γ are at rational coordinates [23]. We can then find an exact (r, t) -threshold-coloring of H for some integers $r \geq 1, t \geq 0$. Scale Γ by a sufficiently large factor t such that each end-point

of some interval in Γ has a positive integer x -coordinate (after possible translation in the positive x direction). Let r be the x -coordinate of the right end-point of the rightmost interval in this scaled representation. Define a coloring $c : V \rightarrow [1 \dots r]$ where for each vertex v of H , $c(v)$ equals the x -coordinate of the left end-point of the interval for v . Also define the threshold as the scaling factor t . It is easy to verify that c is indeed an (r, t) -threshold-coloring.

2.3. Graph Sandwich Problem

The graph sandwich problem is defined in [24] as follows. Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same vertex set V , where $E_2 \subseteq E_1$, and a property Π , does there exist a graph $H = (V, E)$ on the same vertex set such that $E_2 \subseteq E \subseteq E_1$ and H satisfies property Π ?

Here E_1 and E_2 can be thought of as *universal* and *mandatory* sets of edges, with E sandwiched between the two sets. We are interested in a particular property for the graph sandwich problem: “proper interval representability”. A graph satisfies *proper interval representability* if it admits a proper interval representation [24].

Lemma 2. *Let $G = (V, E_G)$ and $H = (V, E_H)$ be two graphs on the same vertex set V such that $E_H \subseteq E_G$. Then the threshold-coloring problem for G with respect to the edge partition $\{E_H, E_G - E_H\}$ is equivalent to the graph sandwich problem for the vertex set V , mandatory edge set E_H , universal edge set $E_H \cup (V \times V - E_G)$ and proper interval representability property.*

PROOF. Let E_U denote the universal edge set $E_H \cup (V \times V - E_G)$ for the graph sandwich problem. Suppose there exists a graph $H^* = (V, E^*)$ such that $E_H \subseteq E^* \subseteq E_U$ and H^* has a proper interval representation. Then by Lemma 1, there exist two integers $r \geq 0$ and $t \geq 0$ and a coloring $c : V \rightarrow [1 \dots r]$ such that for any pair $u, v \in V$, $|c(u) - c(v)| \leq t$ if and only if $(u, v) \in E^*$. We now show that c is in fact a desired threshold-coloring for G . Consider an edge $e = (u, v) \in E_G$. If $e \in E_H$ then $e \in E^*$ since $E_H \subseteq E^*$ and hence $|c(u) - c(v)| \leq t$. On the other hand if $e \in (E_G - E_H)$, $e \notin E_U = E_H \cup (V \times V - E_G)$ and therefore $e \notin E^*$ since $E^* \subseteq E_U$. Hence $|c(u) - c(v)| > t$.

Conversely, if there exist integers $r \geq 1$ and $t \geq 0$ such that there is an (r, t) -threshold-coloring $c : V \rightarrow [1 \dots r]$ of G with respect to the edge partition $\{E_H, E_G - E_H\}$, then define an edge set E^* as follows. For any pair $u, v \in V$, $(u, v) \in E^*$ if and only if $|c(u) - c(v)| \leq t$. Clearly the graph $H^* = (V, E^*)$ has an exact (r, t) -threshold-coloring and hence by Lemma 1, H^* has a proper interval representation. Furthermore for any edge $e = (u, v) \in E_H$, $|c(u) - c(v)| \leq t$ and hence $e \in E^*$. Thus $E_H \subseteq E^*$. Again if $e \in E^*$ then $|c(u) - c(v)| \leq t$. Therefore either $e \in E_H$ or $e \notin E_G \Rightarrow e \in (V \times V - E_G)$. Hence $e \in (E_H \cup (V \times V - E_G)) = E_U$. Thus $E^* \subseteq E_U$. Therefore E^* is sandwiched between the mandatory and the universal set of edges and H^* has a proper interval representation.

The following theorem follows from Observation 1, Lemmas 1 and 2 since the vertex coloring and the graph sandwich problems for proper interval representability are NP-complete [25] and the proper interval recognition can be solved in linear time [20–22].

Theorem 1. *The **Threshold-Coloring** and **Fixed-Threshold-Coloring** problems are NP-complete, while the **Exact-Threshold-Coloring** problem can be solved in linear time.*

3. Total-Threshold-Coloring of Graphs

In this section we address the **Total-Threshold-Coloring** problem: can every spanning subgraph of a graph G be represented by appropriately coloring the vertices of G ?

First note that not every graph (not even every planar graph) is total-threshold-colorable. Suppose that $G = K_4$, and we would like to represent a subgraph where four of the edges remain and span a 4-cycle, while the other two edges are removed. Assume that there exists an (r, t) -threshold-coloring with colors c_1, c_2, c_3, c_4 for vertices v_1, v_2, v_3, v_4 respectively. Without loss of generality assume c_4 is the highest color and $(v_1, v_4) \in F$, hence also $(v_2, v_3) \in F$. Also assume $c_3 \geq c_2$ and consequently $c_4 - c_2 \geq c_3 - c_2$. The left side of the inequality should be at most t , and the right side strictly greater than t , which cannot be accomplished by any choice of the range and the threshold.

On the other hand, for paths and trees there is a simple threshold-coloring with $t = 0$ and two colors. Choose an arbitrary vertex as the root and color it 0. Color 1 all vertices with an odd number of far edges on the shortest path to the root. Color 0 all vertices with an even number of far edges to the root. Then all vertices connected by a near edge of G get the same color, and vertices connected by a far edge get different colors. We thus have the following lemma.

Lemma 3. *Paths and trees are $(2, 0)$ -total-threshold-colorable.*

Next we present a generic algorithm for finding a threshold-coloring of certain graphs; a similar proof is used in [26] in the context of the unit interval representation problem. A *2-independent set* is an independent set I such that the shortest path between any 2 vertices of I has length at least 3.

Lemma 4. *Suppose $G = (I \cup T, E)$ is a graph such that I is 2-independent, the subgraph induced by T is a forest, and I and T are disjoint. Then G is $(5, 1)$ -total-threshold-colorable.*

PROOF. Suppose $\ell : E \rightarrow \{N, F\}$ is an edge-labeling. Let G_T be the forest induced by T . For each $v \in I$, set $c(v) = 0$. Each vertex in T is assigned a color from $\{-2, -1, 1, 2\}$ as follows. Choose a component T' of G_T , and select a root vertex w of T' . If w is far from a neighbour in I , set $c(w) = 2$. Otherwise, $c(w) = 1$. Now we conduct breadth first search on T' , coloring each vertex as it is traversed. When we traverse to a vertex $u \neq w$, it has one neighbour $x \in T'$ which has been colored, and at most one neighbour $v \in I$. If v exists, we choose the color $c(u) = 1$ if $\ell(u, v) = N$, and $c(u) = 2$ otherwise. Then, if the edge (u, x) is not satisfied, we multiply $c(u)$ by -1 . If v does not exist, we choose $c(u) = 1$ or -1 to satisfy the edge (u, x) . By repeating the procedure on each component of G_T , we construct a $(5, 1)$ -threshold-coloring of G with respect to the labeling ℓ .

There is a trivial decomposition into a 2-independent set and a forest of a cycle and a fan (a graph obtained from a path by adding a new vertex connected to all vertices of the path). Thus, by Lemma 4 we have

Theorem 2. *Paths, cycles, trees, and fans are $(5, 1)$ -total-threshold-colorable.*

We notice that all our examples of non-threshold-colorable graphs (e.g., K_4) have short cycles, which can be used to force groups of vertices to be simultaneously near and far. Next we show that planar graphs without short cycles are always total-threshold-colorable.

Theorem 3. *Let G be a planar graph without cycles of length less than or equal to 9. Then G is $(8, 2)$ -total-threshold-colorable.*

The outline of our proof for Theorem 3 is as follows. We first find some small tree structures T that are “reducible”, in the sense that for any edge-labeling of T and any given fixed coloring of the leaves of T to the colors $\{0, 1, \dots, 7\}$, there is an $(8, 2)$ -threshold-coloring of T . For a contradiction assume that there is a planar graph with girth at least 10 having no $(8, 2)$ -threshold-coloring. We consider a minimal such graph G , and by a discharging argument prove that G contains at least one of these reducible tree structures. This contradicts the minimality of G . We start with some technical lemmas.

Extending a coloring. Let P_n be a path with vertices v_0, \dots, v_n . Given an edge-labeling of P_n and the color c_0 of v_0 we call a color c_n *legal* if there exists a $(8, 2)$ -threshold-coloring c of P_n , so that $c(v_0) = c_0$ and $c(v_n) = c_n$.

Lemma 5. *Let P_1 be a path of length 1. Then at least one of the colors 1 or 6 is legal (irrespective of the edge label and the color c_0).*

PROOF. One only needs to observe that color 1 is close to 0, 1, 2, 3, and is far from 4, 5, 6, 7, that is, the distance between colors is at most 2 or strictly more than 2, respectively. The result follows by symmetry.

Lemma 6. *Let P_2 be a path of length 2. Then 3 is legal unless $c_0 = 3$ and $\{N, F\} = \{\{e_1\}, \{e_2\}\}$, that is, the edges e_1 and e_2 are labeled differently. Symmetrically, 4 is legal unless $c_0 = 4$ and $\{N, F\} = \{\{e_1\}, \{e_2\}\}$.*

PROOF. By symmetry we only give the proof for the case $c_2 = 3$. If $N = \{e_1, e_2\}$ then we choose $c(v_1)$ to be the average of c_0 and c_2 , rounding if necessary. If $F = \{e_1, e_2\}$, then one of 0 or 7 is a good choice for $c(v_1)$, as both 0, 7 are far from $c_2 = 3$, and at least one is far from c_0 . In the remaining case we may assume that $c_0 \neq 3$. If $c_0 < 3$, then set $c(v_1) = 0$ or $c(v_1) = 5$ in case $e_2 \in F$ or $e_2 \in N$, respectively. If $c_0 > 3$, then set $c(v_1) = 6$ or $c(v_1) = 1$ in case $e_2 \in F$ or $e_2 \in N$, respectively.

Lemma 7. *Let P_3 be a path of length 3. Then 1, 3, 4, and 6 are all legal (irrespective of the edge label and the color c_0).*

PROOF. By symmetry it is enough to find appropriate coloring extensions for which $c(v_3) = 1$ and $c(v_3) = 3$. For the latter, choose $c_1 = c(v_1) \neq 3$, according to c_0 and the label of e_1 . Now by Lemma 6 this choice of c_1 can be extended to the remaining part of P_3 , so that $c(v_3) = 3$. The goal $c(v_3) = 1$ splits into two subcases. If $c_0 \neq 3, 4$, then by Lemma 6 both 3 and 4 are possible color choices for $c(v_2)$. One is close and the other is far from 1. In case c_0 is either 3 or 4, then again by Lemma 6 both 1 and 6 are possible choices for $c(v_2)$. Again, the former is close and the latter is far from 1.

We call the graph $K_{1,n}$, $n \geq 3$ a *star*. A *spider* is any subdivision² of a star, and its *center* is the single vertex with degree greater than 2. Let T be a spider. A *prong* of T is a path from a leaf to the center of T . We call a prong with k edges a k -prong, and we say that it has length k .

Lemma 8. *Let T be a subdivision of $K_{1,3}$ with prongs of length 1, 2, and 3, respectively. Assume that the leaves of T are assigned colors, so that the leaf u on the 1-prong is colored with either 1 or 6. Then we can extend this partial coloring to all of T .*

PROOF. Let v be the center of T . Given $c(u)$, we can choose $c(v) \in \{3, 4\}$ so that the labeling condition on the 1-prong is satisfied. If this choice cannot be extended to the longer prongs, then the leaf of the 2-prong is also colored with either 3 or 4, see Lemma 6. But then the choice $c(v) \in \{1, 6\}$ which satisfies the labeling condition on the 1-prong can be extended to the remaining prongs.

Reducible configurations. A *configuration* is a tree T , and is *reducible* if every assignment of colors to the leaves of T can be, for every possible edge-labeling of T , extended to a $(8, 2)$ -threshold-coloring c of the whole T .

Lemma 9. *A path P_4 of length 4 is a reducible configuration.*

PROOF. Let v be a neighbor of a leaf in P_4 . By Lemmas 5 and 7 either $c(v) = 1$ or $c(v) = 6$ extends to the remaining uncolored vertices.

Now Lemma 9 implies that longer paths are reducible as well. Let us turn our attention to spiders.

Lemma 10. (A) *Let T be a spider with at most 1 prong of length 1 and the remaining prongs have length 3. Then T is reducible.*
 (B) *Let T be a spider with at most 3 prongs of length 2 and the remaining prongs have length 3. Then T is reducible.*

PROOF. In both cases let v denote the center of the spider. In order to establish (A) let $c(v)$ be either 1 or 6, which is appropriate for the 1-prong (such a choice exists by Lemma 5). By Lemma 7 the coloring $c(v)$ can be extended to the remaining 3-prongs.

²In a *subdivision* of an edge (u, v) in a graph G the edge (u, v) is replaced with a path $ux_1x_2 \dots x_pv$, where each x_i , $i \in \{1, 2, \dots, p\}$ has degree 2. A *subdivision* of a graph G is another graph obtained by subdividing some edges of G .

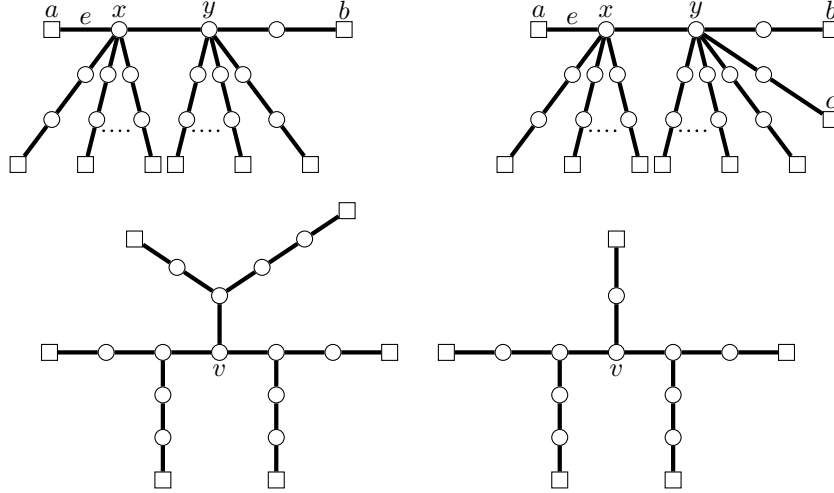


Figure 2: Additional types of reducible configurations, $T1$, $T2$, $T3$, and $T4$ from top left to bottom right.

For (B) we may assume that neither 3 nor 4 can be extended to all three 2-prongs. By Lemma 6 both colors 3 and 4 are used at leaves of the 2-prongs. Now, by Lemma 5 at least one of $c(v) = 1$ or $c(v) = 6$ extends to the third 2-prong, and hence also to the remaining 2- and 3-prongs, by Lemmas 6 and 7.

Two configurations $T1$ and $T2$ are shown in the top row of Figure 2. Each is composed of a path $P = (a, x, y, z, b)$. In $T1$, deleting the vertices of P other than x (respectively, y) causes x (y) to become the center of a spider whose prongs are of length 3. $T2$ differs only in that the spider with center y has an additional prong of length 2. In the figure, a circle vertex has only those edges shown, while a square vertex may have edges connecting it to the other vertices of G .

Lemma 11. $T1$ and $T2$ are reducible configurations.

PROOF. Let us first consider the $T1$ configuration. By Lemma 5 one of 1,6 is appropriate for the color of x , with respect to color $c(a)$ and type of edge e . If $c(b) \in \{3, 4\}$, then choose $c(y)$ from 1, 6, and if $c(b) \notin \{3, 4\}$ then choose $c(y)$ from $\{3, 4\}$. By Lemma 6 this works.

Let us now turn to $T2$. If e is a near edge, we might as well contract e (which implies both x and y will receive the same color), and reduce to Lemma 10(B).

Hence we shall assume e is a far edge. By Lemma 7, coloring x with 1 or 6 extends to the 3-prongs of x . We consider only the case $c(x) = 1$; the case that $c(x) = 6$ is symmetric. If $c(x) = 1$ and $c(y) = 4$ does not extend to the right 2-prongs at y , we may assume $c(b) = 4$. If $c(x) = 6$ and $c(y) = 3$ does not extend to the right 2-prongs at y , we may assume $c(c) = 3$. In this case setting $c(x) = 1$ and $c(y) = 6$ extends to the right.

| | | | | | | | |
|------------------------------|----|---|---|----|----|----|-----|
| degree $\deg(v)$ | 2 | 3 | 4 | 5 | 6 | 7 | ... |
| initial charge $\gamma_0(v)$ | -2 | 2 | 6 | 10 | 14 | 18 | ... |

Table 2: Initial charges of vertices by degree in the proof of Theorem 3.

Two more reducible configurations $T3$ and $T4$ are shown in the bottom roll of Figure 2.

Lemma 12. *The $T3$ and $T4$ are reducible configurations.*

PROOF. By Lemma 8, assigning $c(v)$ either 1 or 6 extends to the remaining vertices of $T3$. For $T4$, the again assigning $c(v)$ either 1 or 6 extends to the remaining vertices by Lemma 5 and Lemma 8.

Discharging. A *minimal counterexample* is a smallest possible (in terms of order) planar graph G without cycles of length less than or equal to 9 which is not $(8, 2)$ -total-threshold-colorable. A minimal counterexample G cannot contain reducible configurations. Further G is connected and has no vertices of degree 1. G is also not a cycle (such a cycle should be of length at least 10 and should not contain a P_4), and is therefore a minor of a graph H of minimal degree at least 3 (since for any vertex of degree 2 in H , one of its incident edges can be contracted to form another graph H' which has fewer degree-2 vertices and has G as a minor).

Let us fix its planar embedding determining its set of faces $F(G)$. Let us define *initial charges*: initial charge of a vertex v , $\gamma_0(v)$, is equal to $4 \deg(v) - 10$, and the initial charge of a face f , $\gamma_0(f)$, is equal to $\deg(f) - 10$. A routine application of Euler formula shows that the total initial charge is -20 .

As all faces have length at least 10, every face is initially non-negatively charged. We shall not alter the charges of faces.

Table 2 shows the initial charges of vertices according to their degree:

The discharging procedure will run in two phases, by $\gamma_i(v)$ we shall denote the charge of vertex v after Phase i of discharging. Informally, Phase 1 shall see that vertices of degree 2 do not have negative charges, and Phase 2 will leave only vertices of degree 3 with a possible negative charge.

Let u, v be vertices of G . We say that u and v are 2-adjacent, if G contains a $u - v$ -path whose (possible) internal vertices all have degree 2. In Phase 1 we redistribute charge according to the following rule:

Rule 1: Every vertex v of degree at least 3 sends charge 1 to every vertex u of degree 2, for which v and u are 2-adjacent.

In Phase 2 we shall apply the following rule:

Rule 2: If u and v are adjacent with $\gamma_1(u) > 0, \gamma_1(v) < 0$ then u sends charge 1 to v .

As every vertex u of degree 2 (we also call them *2-vertices*) is 2-adjacent to exactly two vertices of bigger degree, we have $\gamma_1(u) = 0$ in this case. For a vertex v of degree at least 3, the discharging in Phase 1 decreases the charge of v by the number of 2-vertices which are 2-adjacent to v .

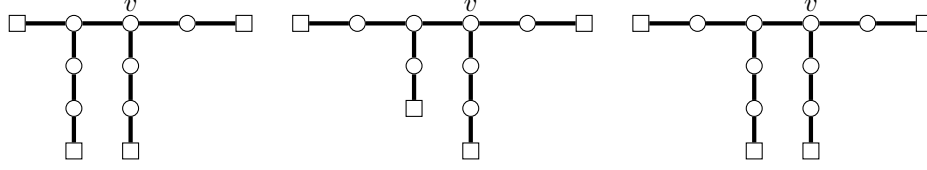


Figure 3: Negatively charged vertex v after both phases induces a reducible configuration.

Let v be a vertex of degree at least 3. A *prong* at v is a $v - x$ -path whose other end-vertex x is of degree at least 3 and has internal vertices of degree 2.

Lemma 13. *Let v be a vertex of degree at least 3. Then the number of 2-vertices that are 2-adjacent to v is at most $2 \cdot \deg(v) - 3$.*

PROOF. By Lemma 9 each prong at v contains at most two vertices of degree 2. If the shortest prong at v has length 1, then Lemma 10 implies that at least one other prong has length ≤ 2 . If the shortest prong at v has length 2, then by Lemma 10 we have at least four prongs that are of length ≤ 2 , and the result follows.

Now Lemma 13 serves as the lower bound for vertex charges after Phase 1, and in turn prepares us for the Phase 2 of discharging.

Lemma 14. (A) *Let v be a vertex of degree 3. If $\gamma_1(v) < 0$, then $\gamma_1(v) = -1$ and the prongs at v have lengths 1, 2 and 3, respectively.*

(B) *Let v be a vertex of degree 3. If $\gamma_1(v) = 0$, then the prongs at v have either lengths 1, 1, 3 or 1, 2, 2.*

(C) *Let v be a vertex of degree 3 with its prongs of length 1, 1, and 2. Then $\gamma_1(v) = 1$.*

(D) *Let v be a vertex of degree 3 with all 3 prongs of length 1. Then $\gamma_1(v) = 2$.*

(E) *If v is a vertex of degree at least 4, then $\gamma_2(v) \geq 0$, and also $\gamma_2(v)$ is not smaller than the number of 1-prongs at v .*

PROOF. Let us first prove (E). Choose a vertex v with $\deg(v) \geq 4$. For every prong of length 3, v sends 2 units of charge in Phase 1. For every shorter prong v sends at most 1 unit of charge in either Phase 1 or Phase 2. The total charge sent out of v in both of the phases is by Lemma 10 and Lemma 13 at most $2 \deg(v) - 2$. Hence $\gamma_2(v) \geq (4 \deg(v) - 10) - (2 \deg(v) - 2) = 2 \deg(v) - 8 \geq 0$.

The other cases merely stratify vertices of degree 3 according to the number of their 2-neighbors of degree 2.

Lemma 14(E) states that every vertex v of degree at least 4 satisfies $\gamma_2(v) \geq 0$. Similarly, if a 3-vertex u is adjacent to a vertex v whose degree is at least 4, then also $\gamma_2(u) \geq 0$. This fact follows from either Lemma 14(A) and (E) (in case $\gamma_1(u) < 0$), or from either Lemma 14(C) or (D) (if $\gamma_1(v) > 0$) as in this case u cannot send excessive charge in Phase 2.

Lemma 15. *No vertex v has $\gamma_2(v) < 0$ and $\gamma_1(v) < 0$.*

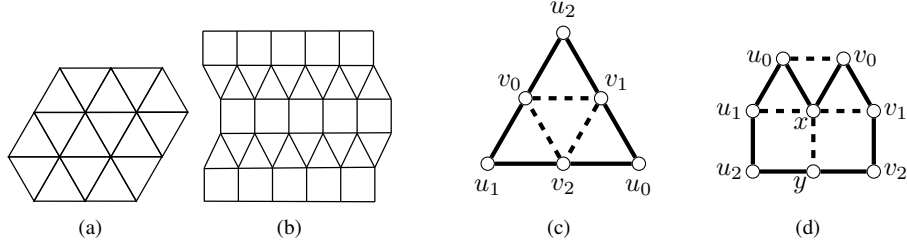


Figure 4: The (a) triangular and (b) square-triangle grids. (c-d) Graphs which are not total-threshold-colorable. Dashed edges are labeled F.

PROOF. Let v be a vertex satisfying both $\gamma_2(v) < 0$ and $\gamma_1(v) < 0$. By Lemma 14 $\deg(v) = 3$ and v has prongs of length 1, 2, 3. Let u be the only neighbor of v of degree not equal to 2. Since v has received no charge from u in Phase 2 we have both $\deg(u) = 3$ and $\gamma_1(u) \leq 0$. By Lemma 14 the prongs of u are of lengths 1, 2, 3 or 1, 1, 3 or 1, 2, 2. Hence G contains one of configurations shown in Figure 3. Now observe that these are reducible, as each matches one of $T1$ or $T2$ types of reducible configurations Lemma 11.

Lemma 16. *No vertex v has $\gamma_2(v) < 0$ and $\gamma_1(v) \geq 0$.*

PROOF. If $\gamma_1(v) = 0$, then also $\gamma_2(v) = 0$, as Rule 2 does not reduce charge of a discharged vertex. By Lemma 14(E) vertices of degree at least 4 do not have negative charge after Phase 2.

Hence we may assume that v has degree 3, $\gamma_1(v) > 0$, and $\gamma_2(v) < 0$. By Lemma 14(C) and (D) every neighbor u of v satisfies either $\deg(u) = 2$ or $\deg(u) = 3$ and $\gamma_1(u) < 0$. The only two possible cases are $T3$ and $T4$, which are reducible by Lemma 12.

Lemmas 15 and 16 imply that no vertex has negative charge after Phase 2 of the discharging procedure. As the total charge remains negative and the faces cannot have negative charges, we have a contradiction, which completes the proof of Theorem 3.

4. Total-Threshold-Coloring of Regular Graphs

In this section, we consider total-threshold-colorability of regular graphs. We prove that some of them are total-threshold-colorable, some are non-total-threshold-colorable, and for the square grid, no constant range of colors suffices.

4.1. Non-Total-Threshold-Colorable Grids

We first show that 2 infinite grids, the triangular and square-triangle grids, are non-total-threshold-colorable; see Figure 4(a)-(b) for an example of each grid.

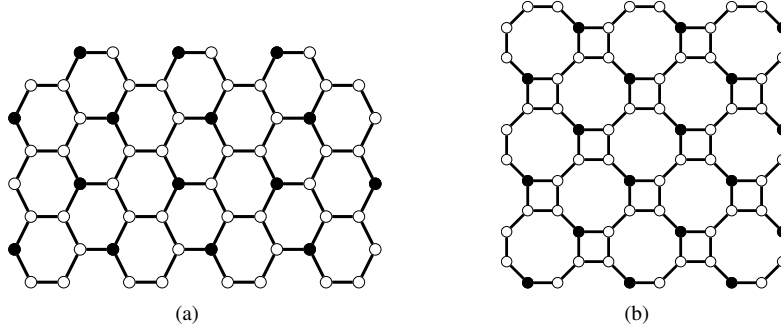


Figure 5: Decomposition of the (a) hexagonal grid and (b) octagonal-square grid into a 2-independent set (white vertices) and forest (black vertices).

Triangular Grid. In a triangular grid (planar weak dual of a hexagonal grid) all faces are triangles and internal vertices have degree 6. It is easy to show that a triangular grid is not total-threshold-colorable. Consider the graph with vertices $v_0, v_1, v_2, u_0, u_1, u_2$, where each vertex u_i is adjacent to v_{i+1} and v_{i+2} (indices modulo 3); see Figure 4(c). Let $F = \{(v_0, v_1), (v_1, v_2), (v_2, v_0)\}$, and let N contain the remaining 6 edges. Assume that there exists an (r, t) -threshold-coloring c . Without loss of generality, let $c(v_0) < c(v_1) < c(v_2)$. Now on one hand $c(v_2) - c(v_0) > 2t$ and on the other $c(v_2) - c(v_0) \leq |c(v_2) - c(u_1)| + |c(u_1) - c(v_0)| \leq 2t$, which is impossible. This also proves that outerplanar graphs are not total-threshold-colorable in general.

Square-Triangle Grid. We prove that the graph in Figure 4(d) is not total-threshold-colorable. Assume to the contrary that c is an (r, t) -threshold-coloring. Without loss of generality let $c(v_0) < c(u_0)$. Since (v_1, u_0) is a far edge and $(v_0, x), (u_0, x)$ are near we have $c(v_0) < c(x) < c(u_0)$. Similar argument shows that $c(v_1) < c(v_0) < c(x) < c(u_0) < c(u_1)$. Then if $x < y$, we have $c(v_1) + t < c(x)$ and $c(x) + t < c(y)$, which implies $c(v_1) + 2t < c(y)$. This makes it impossible to find a color for v_2 near to both v_1 and y . Similarly if $x > y$ then it is impossible to color u_2 .

Theorem 4 summarizes the results.

Theorem 4. *The triangle grid and triangle-square grid are not total-threshold-colorable.*

4.2. Total-Threshold-Colorable Grids

Hexagonal and Octagonal-Square Grids. In the hexagonal grid (planar weak dual of the triangular grid) all faces are 6-sided and internal vertices have degree 3. The octagonal-square grid contains 8-sided and 4-sided faces and internal vertices of degree 3. It is easy to see that the grids admit a decomposition into a 2-independent set and a forest; see Figure 5. Hence, the grids are colorable by Lemma 4.

Theorem 5. *Any hexagonal grid and octagonal-square grid is $(5, 1)$ -total-threshold-colorable.*

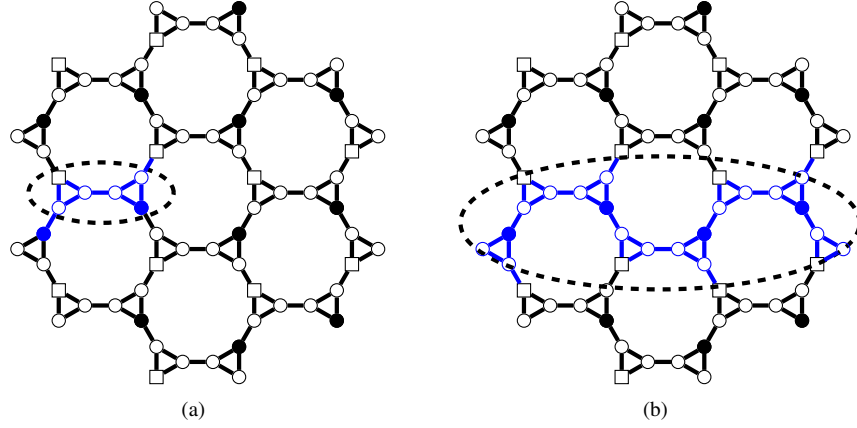


Figure 6: Threshold-coloring of the triangle-dodecagon grid: (a) one patch has been colored, shown inside the oval; (b) coloring an entire row.

Triangle-Dodecagon and Square-Hexagon-Dodecagon Grids. In order to color the graphs, we use $t = 2$ and $r = 9$ colors such as $\{0, \pm 1, \pm 2, \pm 3, \pm 4\}$. This color-space has the following properties.

Lemma 17. Consider a path with 3 vertices (v_0, v_1, v_2) , such that v_0, v_2 have colors $c(v_0)$ in $\{0, \pm 1\}$ and $c(v_2)$ in $\{\pm 1, \pm 2, \pm 3, \pm 4\}$. For threshold 2 and any edge labeling,

- (a) If $c(v_0) = 0$, and $c(v_2) \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$, then we can choose $c(v_1)$ in $\{\pm 2, \pm 3\}$.
- (b) If $c(v_0) = 0$ and $c(v_2) \in \{\pm 2, \pm 3, \pm 4\}$, then we can choose $c(v_1)$ in $\{\pm 2, \pm 4\}$.
- (c) If $c(v_0) = \pm 1$, and $c(v_2) \in \{\pm 2, \pm 3\}$, then we can choose $c(v_1)$ in $\{\pm 1, \pm 4\}$.

PROOF. (a) First, we choose $c(v_1) = \pm 2$ if v_1 is near to v_0 , and ± 3 otherwise. Then, if v_1 is near to v_2 , choose the sign of $c(v_1)$ to agree with $c(v_2)$. Otherwise choose the sign of $c(v_1)$ to be opposite $c(v_2)$.

(b) Choose $c(v_1) = \pm 2$ if v_1 is near to v_0 , and ± 4 otherwise. Then, choose the sign of $c(v_1)$ as before.

(c) Choose $c(v_1) = \pm 1$ if v_1 is near to v_0 , and $c(v_1) = \pm 4$ otherwise. Then, choose the sign of $c(v_1)$ as before.

On a high level, our algorithms for both grids, are very similar to each other: we identify small “patches”, and then assemble them into the grid; see Figures 7-6. We first show how to color a patch for the triangle-dodecagon grid.

Lemma 18. Let G be the graph shown in Figure 7(a). Suppose $c(u_0) = c(u_1) = 0$ and $c(v_0) = \pm 1$. Then for any edge-labeling, this coloring can be extended to a $(9, 2)$ -threshold-coloring of G such that v_5 is colored 1 or -1 .

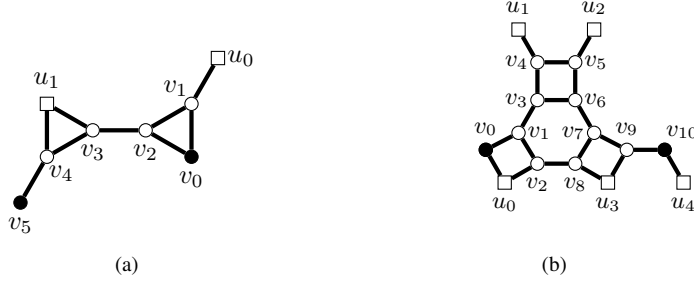


Figure 7: Illustration of Lemma 18 and 19. (a) A subgraph of the triangle-dodecagon grid. (b) A subgraph of the square-hexagon-dodecagon grid. Square vertices are colored 0.

PROOF. Assume $c(v_0) = 1$. We apply Lemma 17(a) to the path (u_0, v_1, v_0) to choose a color for v_1 in $\{\pm 2, \pm 3\}$, then apply part (c) of the lemma to the path (v_0, v_2, v_1) to choose $c(v_2) \in \{\pm 1, \pm 4\}$. Then $c(v_3)$ is chosen in $\{\pm 2, \pm 3\}$ using part (a) of the lemma on the path (u_1, v_3, v_2) , and finally $c(v_4) \in \{\pm 2, \pm 3\}$ is chosen using part (a) on the path (u_1, v_4, v_3) . Then we may choose $c(v_5) = 1$ or -1 so that it is near or far from $c(v_4)$.

A similar lemma applies to the square-hexagon-dodecagon grid.

Lemma 19. *Let G be the graph shown in Figure 7(b), and consider any edge-labeling. Suppose that $c(u_i) = 0$, for $i = 0, \dots, 4$, and $c(v_0)$ is a fixed color in $\{\pm 2, \pm 4\}$ that satisfies the label of (v_0, u_0) . Then we can extend this partial coloring to a coloring c of all of G , so that c is a $(9, 2)$ -threshold-coloring of G with respect to the edge-labeling, and $c(v_{10})$ is in $\{\pm 2, \pm 4\}$.*

PROOF. Let ℓ be an edge-labeling of G . We consider only the case where $c(v_0) \in \{2, 4\}$ as the other case is symmetric. First, let $c(v_6) = 1$. Using Lemma 17, we color v_5, v_4 , and v_3 so that $c(v_3)$ is in $\{\pm 1, \pm 4\}$. Consider Table 3, where we list valid colors of v_1 in $\{\pm 2, \pm 4\}$ according to edge-labeling and $c(v_3)$. An “x” indicates no color can be chosen, but in these cells we multiply $c(v_3)$ by -1 to obtain a color for v_1 , and we multiply $c(v_4), c(v_5)$, and $c(v_6)$ by -1 so that this is consistent. Use Lemma 17 to choose colors for v_2, v_8, v_7, v_9 , and v_{10} so that $v_{10} \in \{\pm 2, \pm 4\}$.

Theorem 6. *The triangle-dodecagon and square-hexagon-dodecagon grids are $(9, 2)$ -total-threshold-colorable.*

PROOF. We prove the lemma for the triangle-dodecagon grid. First, we join several copies of the graph G in Lemma 18. Let G_1, \dots, G_n be copies of G . Let us call $u_{i,k}$ and $v_{j,k}$ the vertices in G_k , corresponding to u_i, v_j ($i = 0$ or $1, 0 \leq j \leq 5$). For $1 \leq k < n$, we set $v_{5,k} = v_{0,k+1}$. This defines a single row of the triangle-dodecagon grid. We can construct a $(9, 2)$ -threshold-coloring of this chain of G_1, \dots, G_n by giving the vertex $v_{0,1}$ the color 1 and repeatedly applying Lemma 18.

| | $\ell(v_0, v_1) = N$ $\ell(v_1, v_3) = F$ | $\ell(v_0, v_1) = N$ $\ell(v_1, v_3) = N$ | $\ell(v_0, v_1) = F$ $\ell(v_1, v_3) = N$ | $\ell(v_0, v_1) = F$ $\ell(v_1, v_3) = F$ |
|---------------|--|--|--|--|
| $c(v_3) = 1$ | 4 | 2 | × | -4,-2 |
| $c(v_3) = -1$ | 2,4 | × | -2 | -4 |
| $c(v_3) = 4$ | × | 2,4 | × | -4,-2 |
| $c(v_3) = -4$ | 2,4 | × | -4,-2 | × |

Table 3: Possible colors for v_1 , given the labeling of (v_0, v_1) and (v_1, v_3) in the proof of Lemma 19. A \times indicates no color can be chosen.

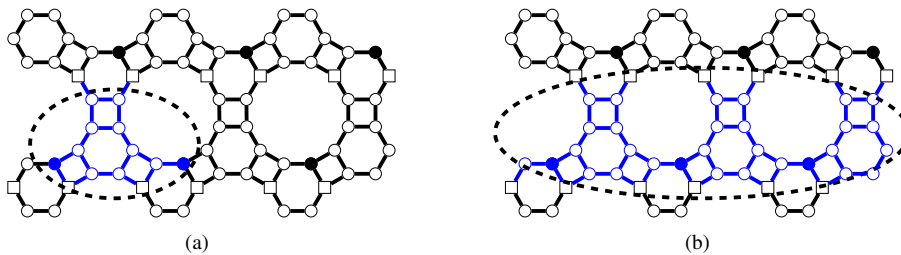


Figure 8: Threshold-coloring of the square-hexagon-dodecagon grid with the patches from Lemma 19. Observe that there are alternating “rows” separated by square vertices. (a) One patch has been colored, shown inside the oval. (b) Extending the coloring to an entire row.

To construct the next row, we add a copy of G connected to G_i and G_{i+2} for each odd i with $1 \leq i \leq k - 2$, by identifying $u_{1,i} = u_0$ and $u_{0,i+2} = u_1$. We then join the copies of G added above the first row in the same way that the copies G_1, \dots, G_n were joined. By repeatedly adding new rows, we complete the construction of the triangle-dodecagon grid. We can threshold-color each row, and since the rows are connected only by vertices colored 0, the entire graph is $(9, 2)$ -total-threshold-colorable; see Figure 6.

The proof of the other grid is identical, and is illustrated in Figure 8.

4.3. The Square Grid

Whether every subgraph of the square grid is total-threshold-colorable is an open question. Here we show that arbitrarily large square grids will require an arbitrarily large range of colors to threshold-color. First, a technical lemma.

Lemma 20. Consider a 4-cycle $(v_0, v_1, v_2, v_3, v_0)$ with edge-labeling ℓ and an (r, t) -threshold-coloring c .

1. Suppose $\ell(v_0, v_3) = \ell(v_2, v_3) = F$ and $\ell(v_0, v_1) = \ell(v_1, v_2) = N$. Then $c(v_0) < c(v_3)$ if and only if $c(v_1) < c(v_3)$ and $c(v_2) < c(v_3)$.
2. Suppose $\ell(v_0, v_1) = \ell(v_2, v_3) = F$ and $\ell(v_0, v_3) = \ell(v_1, v_2) = N$. Then $c(v_0) < c(v_1)$ if and only if both of $c(v_0), c(v_3)$ are less than both of $c(v_1), c(v_2)$.

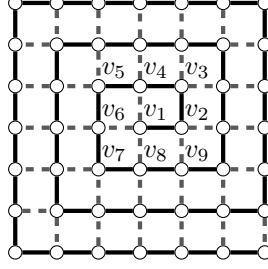


Figure 9: An example of a square grid requiring a large number of colors. Dashed edges are far.

PROOF. In both cases, we prove the implication in only one direction, since the other direction is symmetric, by considering $>$ instead of $<$ (since the edges are F , we cannot have equality in any of the considered cases).

1. Suppose that $c(v_0) < c(v_3)$ but $c(v_1) \geq c(v_3)$ or $c(v_2) \geq c(v_3)$. Then $c(v_0) < c(v_3) - t$ and $|c(v_0) - c(v_1)| \leq t$, so $c(v_1) < c(v_3)$. Therefore $c(v_2) < c(v_3) + t$, so $c(v_2)$ must be less than $c(v_3)$ since $|c(v_2) - c(v_3)| > t$.
2. Suppose that $c(v_0) < c(v_1)$. Then $c(v_0) < c(v_1) - t$, $c(v_2) \geq c(v_1) - t$, and so $c(v_1) < c(v_2)$. We have $c(v_3) < c(v_1)$ since $|c(v_0) - c(v_3)| \leq t$. If $c(v_3) > c(v_2)$, then $c(v_1) - t \leq c(v_2) < c(v_3) < c(v_1)$, so $|c(v_2) - c(v_3)| \leq t$, a contradiction.

Theorem 7. For every $r > 0$, there exist finite subgraphs of the square grid, which are not (r, t) -total-threshold-colorable for any $t \geq 0$.

PROOF. Let S be the infinite square grid, drawn as in Figure 9. A vertex v in S has north, east, south, and west neighbors. If $P = (v_1, \dots, v_j)$ is a path in S , we call P a *north path* if v_{i+1} is the north neighbour of v_i for all $1 \leq i < j$. *East, south, and west paths* are defined similarly and these paths are uniquely defined for a given start v_i and number of vertices j .

For each odd $n > 0$, we define a path $S_n = (v_1, \dots, v_{n^2})$ in S . Let S_1 be the path consisting of a single chosen vertex v_1 of S . Let $k = n + 2$, and recursively construct S_k from S_n by first adding the east neighbour v_{n^2+1} of v_{n^2} to S_n . Then, we add the north path $(v_{n^2+1}, \dots, v_{n^2+k})$, the west path $(v_{n^2+k}, \dots, v_{n^2+2k})$, the south path $(v_{n^2+2k}, \dots, v_{n^2+3k})$, and the east path $(v_{n^2+3k}, \dots, v_{n^2+4k})$; see Figure 9.

With S_n defined for every odd n , let $G_n = (V_n, E_n)$ be the subgraph of S induced by the vertices of S_n , and let $\ell_n : E_n \rightarrow \{N, F\}$ be an edge-labeling such that $\ell_n(e) = N$ if and only if e is in S_n . The graph G_7 is shown in Figure 9. We now prove that G_n requires at least n colors to threshold-color, for any threshold $t > 0$. W.l.o.g. suppose that c is a threshold-coloring such that $c(v_4) > c(v_1)$. Note that the cycles (v_4, v_5, v_6, v_1) , (v_6, v_7, v_8, v_1) , and (v_8, v_9, v_2, v_1) match the cycles in Lemma 20, implying that $c(v_6), c(v_8)$ and $c(v_9)$ are greater than $c(v_1)$. This serves as the basis for induction. Suppose that for some odd $k > 1$, the vertex

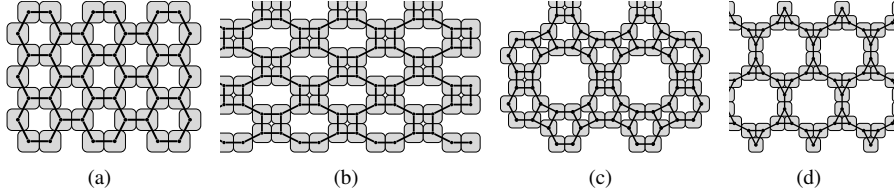


Figure 10: Unit-cube contact representations for hexagonal, octagonal-square, square-hexagon-dodecagon, and triangle-dodecagon grids. Corners are rounded to emphasize proper contact.

$c(v_{k^2}) > c(v_{(k-2)^2})$ for any assignment c of colors to the vertices of G_n , so long as $c(v_4) > c(v_1)$ and c is an (r, t) -threshold-coloring for some $r > 0$. Then we consider the color $c(v_i)$, for $k^2 < i \leq (k+2)^2$. There are three cases. In the first, v_i is the interior vertex of a north, east, west, or south path in S_{k+2} . Then v_i is on a cycle $(v_{i-1}, v_i, v_j, v_{j-1})$, $j \leq k^2$, with $\ell_n(v_i, v_{i-1}) = \ell_n(v_j, v_{j-1}) = N$ and $\ell_n(v_i, v_j) = \ell_n(v_{i-1}, v_{j-1}) = F$. By Lemma 20, we have $c(v_i) > c(v_j)$ and $c(v_i) > c(v_{j-1})$ so long as $c(v_{i-1}) > c(v_{j-1})$. In the second case, v_i is part of a 4-cycle $(v_{i-1}, v_i, v_{i+1}, v_j)$, $j \leq k^2$, with $\ell_n(v_{i-1}, v_i) = \ell_n(v_i, v_{i+1}) = N$, and the other edges labeled F . Again by Lemma 20, we have $c(v_i) > c(v_j)$ and $c(v_{i+1}) > c(v_j)$ so long as $c(v_{i-1}) > c(v_j)$. The third case is the same, except v_i is in the place of v_{i+1} .

Given these three cases and the assumption that $c(v_{k^2}) > c(v_{(k-2)^2})$, we conclude that $c(v_{(k+2)^2}) > c(v_{k^2})$ for each odd $k > 1$. Therefore, the graph G_n , with edge-labeling ℓ_n , requires a distinct color for each of $c(v_1), c(v_{3^2}), \dots, c(v_{n^2})$.

5. Unit-Cube Contact Representations of Graphs

Lemma 21. *If G has a unit-cube contact representation Γ so that one face of each cube is co-planar in Γ , then any threshold subgraph of G also has a unit-cube representation.*

PROOF. Let $H = (V, E_H)$ be a threshold subgraph of $G = (V, E_G)$ and let $c : V \rightarrow [1 \dots r]$ be an (r, t) -threshold-coloring of G with respect to the edge-labeling defined by H . We now compute a unit-cube contact representation of H from Γ using c .

Assume (after possible rotation and translation) that the bottom face for each cube in Γ is co-planar with the plane $z = 0$; see Figure 1(a). Also assume (after possible scaling) that each cube in Γ has side length $t + \epsilon$, where $0 < \epsilon < 1$. Then we can obtain a unit-cube contact representation of H from Γ by lifting the cube for each vertex v by an amount $c(v)$ so that its bottom face is at $z = c(v)$; see Figure 1(b). Note that for any edge $(u, v) \in E_H$, the relative distance between the bottom faces of the cubes for u and v is $|c(u) - c(v)| \leq t < (t + \epsilon)$; thus the two cubes maintain contact. On the other hand, for each pair of vertices u, v with $(u, v) \notin E_H$, one of the following two cases occurs: (i) either $(u, v) \notin E_G$ and their corresponding cubes remain non-adjacent as they were in Γ ; or (ii) $(u, v) \in (E_G - E_H)$ and the relative distance between the bottom faces of the two cubes is $|c(u) - c(v)| \geq (t + 1) > (t + \epsilon)$, making them non-adjacent.

Theorem 8. *Any subgraph of the hexagonal, octagonal-square, triangle-dodecagon, and square-hexagon-dodecagon grid has a unit-cube contact representation.*

PROOF. Each of the grids is total-threshold-colorable and has a unit-cube contact representation; see Figure 10(c)–(d).

6. Conclusion and Open Problems

We introduced a new graph coloring problem, called threshold-coloring, that generates spanning subgraphs from an input graph where the edges of the subgraph are implied by small absolute value difference between the colors of the endpoints. We showed that any spanning subgraph of trees, some planar grids, and planar graphs without cycles of length less than or equal to 9 can be generated in this way; for other classes like triangular and square-triangle grids, we showed that this is not possible. We also considered different variants of the problem and noted relations with other well-known graph coloring and graph-theoretic problems. Finally, we use the threshold-coloring problem to find unit-cube contact representation for all the subgraphs of some planar grids. The following is a list of some interesting open problems and future work.

1. Some classes of graphs are total-threshold-colorable, while others are not. There are many classes for which the problem remains open, e.g., the square grid.
2. Planar graphs with cycles of length ≥ 10 are total-threshold-colorable, while all our non-threshold-colorable graphs contain triangles: can this 3-10 gap be reduced?
3. Can we efficiently recognize graphs that are total-threshold-colorable?
4. Is there a good characterization of total-threshold-colorable graphs?
5. The threshold-coloring problem is NP-complete in general. Which restrictions on G and/or H allow it to be polynomial time solvable?

Acknowledgments: Research of Md. Jawaherul Alam, Stephen G. Kobourov, Sergey Pupyrev and Jackson Toeniskoetter supported in part by NSF grants CCF-1115971 and DEB 1053573. We thank Torsten Ueckerdt, Carola Winzen and Michalis Bekos for discussions about different variants of the threshold-coloring problem.

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