Colored Simultaneous Geometric Embeddings and Universal Pointsets*

Alejandro Estrella-Balderrama, J. Joseph Fowler, and Stephen G. Kobourov

[1] Department of Computer Science, University of Arizona, {aestrell,jfowler}@cs.arizona.edu
[2] AT&T Research Labs, skobourov@research.att.com

Technical Report 09-02
May 14, 2009

Abstract. Universal pointsets can be used for visualizing multiple relationships on the same set of objects or for visualizing dynamic graph processes. Using the same point in the plane to represent the same object helps preserve the viewer's mental map. Small universal pointsets are highly desirable but often do not exist because of the restriction that a given object must be mapped to a fixed point in the plane. In colored simultaneous embeddings this restriction is relaxed, by allowing a given object to map to a subset of points in the plane. Specifically, consider a set of graphs on the same set of n vertices partitioned into k colors. Finding a corresponding set of k-colored points in the plane in which each vertex is mapped to a point of the same color so as to allow a straight-line plane drawing of each graph is the problem of *colored simultaneous geometric embedding*. For trees, we show that there exists small universal pointsets (1) for 3-colored caterpillars of size n, (2) for 3-colored radius-2 stars of size $n+3$, and (3) for 2-colored spiders of size n. For outerplanar graphs, we show that these same universal pointsets also suffice for (1) 3-colored K_3-caterpillars, (2) 3-colored K_3-stars, and (3) 2-colored fans, respectively. We also show that there exist (i) a 2-colored planar graph and pseudo-forest, (ii) three 3-colored outerplanar graphs, (iii) four 4-colored pseudo-forests, and (iv) three 5-colored pseudo-forests without simultaneous embeddedings.

1 Introduction

Applications in bioinformatics, social sciences, and software engineering often need to simultaneously visualize a set of related graphs. For instance, in Unified Modeling Language (UML) diagrams there can be many different relationships between the same set of entities. A given class or component can be present in many different types of diagrams. Some diagrams depict internal structures, while others show global interactions, while still others show a combination of both. These interactions are often complex and difficult to visualize in a single diagram and so users need to extract information from multiple diagrams of different types.

As a result, the user has to maintain an internal mental map of the system. In order to navigate a system, preserving the mental map of related structures that serve as landmarks in the different diagrams is vital. To facilitate easier reconstruction of the mental map when the user examines a series of related diagrams, corresponding nodes can be co-located and common edges can be drawn in the same way. To improve readability, edge crossings within each diagram, or layer, are undesirable. However, crossings between edges of different layers is permitted. This leads to the problem of *simultaneous geometric embeddings* [3], which generalizes the notion of planarity to multiple layers.

In this paper, we only consider geometric drawings with straight-line edges. We omit the "geometric" clarification henceforth. There are two variations of simultaneous embeddings: *with* and *without mapping*. In the first, a 1-1 mapping between vertices of different layers is part of the input. Two corresponding vertices in different graphs are then drawn on the same point. In the latter, each vertex of a layer can be placed at any one of the points in the pointset, irrespective of the placement of the vertices in other layers.

The problem of *colored simultaneous embedding* [2] generalizes these two extremes. The input is a set of planar graphs $G_1 = (V,E_1)$, $G_2 = (V,E_2)$, ..., $G_r = (V,E_r)$ all on the *same* set of vertices $|V| = n$ strictly partitioned into *k colors*. That is to say $V = V_1 \cup V_2 \cup \cdots \cup V_k$ where $V_i \cap V_j = \varnothing$ for $1 \le i < j \le k$ in which the vertices of V_i have color c_i for $i \in [1..k]$ in each graph G_j for $j \in [1..r]$.

The output is a set of points $|P| \ge n$ that are also strictly partitioned into k colors (in which each color class of P is at least the size of the corresponding color class of V) with a fixed embedding in the Euclidean plane such that each graph G_j for $j \in [1..r]$ has a straight-line planar drawing where each vertex of color c_i for $i \in [1..k]$ is placed on exactly one point of P also of color c_i. Unless specified otherwise, $|P| = n$.

* This work is supported in part by NSF grants CCF-0545743 and ACR-0222920.

The partitioning of V into k colors gives a partial mapping between graphs. If all $k = n$ colors are used, then it is a 1-1 mapping in which each vertex of V is mapped to precisely one point of P. If only $k = 1$ color is used, then there is no mapping, and each vertex of V can be freely placed on any one point of P. This corresponds to the problems of simultaneous embedding with and without mapping, respectively.

1.1 Previous Work

Brass *et al.* [3] showed that some planar pairs, such as pairs of cycles and pairs of caterpillars (removal of all endpoints leaves a path) always admit a simultaneous embedding with mapping. They also showed that this is not always possible with pairs of outerplanar graphs on 6 vertices, a path and a planar graph on 9 vertices, and three paths also on 9 vertices. More recently, Kaufmann *et al.* [16] showed that simultaneous embedding with mapping do not always exist even for pairs of trees. If no mapping is given, Brass *et al.* [3] also showed that a planar graph and any number of outerplanar graphs share a simultaneous embedding. It is not known whether this is possible for an arbitrary pair of planar graphs.

Simultaneous embedding is also related to *universal pointsets* problems, where the goal is to find a pointset P in the Euclidean plane that allows for any number of graphs of a given class to be drawn with straight-line edges and no crossings on P. Rosenstiehl and Tarjan [17] posed the question of whether there exists a universal pointset of size n for all n-vertex planar graphs. The question was answered in the negative by de Fraysseix *et al.* [4] who presented a set of n-vertex planar graphs that requires a pointset of size $\Omega(n+\sqrt{n})$. For restricted classes of n-vertex planar graphs universal pointsets of size n have been found. Gritzman *et al.* [14] showed that a set of n points in general position is a universal pointset for trees and outerplanar graphs for which Bose [1] gives efficient drawing algorithms.

For the problem of colored simultaneous embedding, Brandes *et al.* [2] proved that any set of 2-colored n points in general position separable by a line in the plane forms a universal pointset of size n for any number of 2-colored paths. This allows the simultaneous embedding of a tree or an outerplanar graph with any number of paths on 2 colors. Brandes *et al.* showed there also exists a universal pointset of size n for 3-colored paths, and provided several negative results in showing that five 5-colored paths, four 6-colored paths, three 6-colored cycles, and three 9-colored paths do not always share a simultaneous embedding.

Relaxing the constraint on the size of the pointset allows for a way to more easily obtain near-simultaneous embeddings, where we attempt to place corresponding vertices relatively close to one another in each drawing. For example, if each clusters of points in the plane has a distinct color, then even if a red vertex v drawn at red point p in G_1 has moved to another red point q in G_2, the movement is limited to the area covered by the red points. This has applications in visualizing dynamic graphs, where the viewer's mental map is preserved by limiting the movement of the vertices [5, 13].

1.2 Our Contribution

We provide universal and near-universal pointsets for three classes of trees: (1) 3-colored caterpillars of size n, (2) 3-colored radius-2 stars (stars, $K_{1,k}$, in which each edge is subdivided at most once) of size $n+3$, in general, and of size $n+1$ in a more restricted case, and (3) 2-colored spiders (stars in which each edge is subdivided arbitrarily) of size n; see Fig. 1(a)–(c). We extend these three universal pointsets to accommodate classes of outerplanar graphs in which every spanning tree is one of the above classes of trees: (1) 3-colored K_3-caterpillars (caterpillars with extra edges (u,w) and (w,v) for any cut-edge (u,v)), (2) 3-colored K_3-stars (stars with an extra edge (u,v) for any pair of leaves u and v), and (3) 2-colored fans (biconnected outerplanar graphs in which all chords are incident to a common vertex); see Fig. 1(d)–(f).

We also show that the following sets of graphs do not always have a colored simultaneous embedding: (i) a pseudo-forest (a graph with at most one cycle) and a planar graph on 2 colors, (ii) three outerplanar graphs on 3 colors, (iii) four pseudo-forests on 4 colors, and (iv) three pseudo-forests on 5 colors.

Note that the only previously known universal pointset for 3-colored graphs was that for paths by Brandes *et al.* [2]. The three types of graphs for which we provide universal pointsets are considerably larger and more involved than paths. Furthermore, taken together, caterpillars, radius-2 stars and degree-3 spiders form

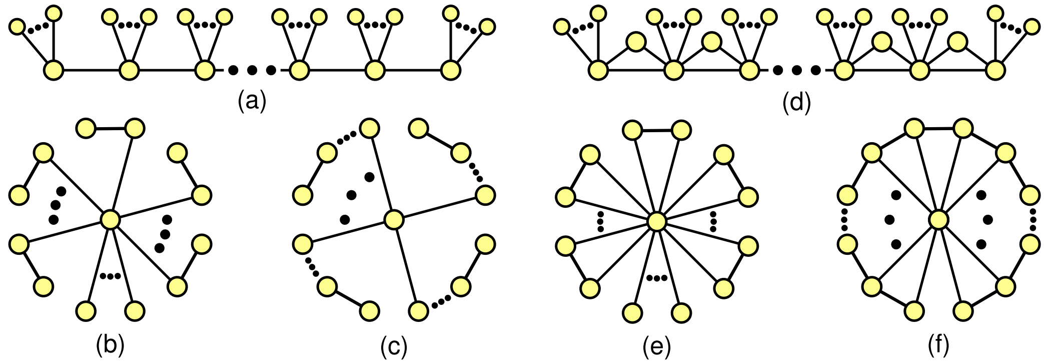

Fig. 1: Three classes of trees: (a) caterpillars, (b) radius-2 stars, and (c) spiders; and three classes of outerplanar graphs: (d) K_3-caterpillars, (e) K_3-stars, and (f) fans.

the unlabeled level planar (ULP) trees, which play a role in minimum level non-planar patterns [15] and simultaneous geometric embeddings and level planarity [8].

2 Universal Pointsets for Trees

We present universal pointsets of cardinality $n+c$, where $c \in \{0,1,3\}$ for three classes of n-vertex trees. The first pointset for 3-colored caterpillars has size n. The second pointset for 3-colored radius-2 stars has size $n+3$, in general, and size $n+1$ in a restricted case, so that not every point is used in every radius-2 star. The last pointset for 2-colored spiders is easily seen to have size $n+2$. By using alternate embedding approaches for roots of different colors, 2 points can be eliminated to get an optimal pointset of size n.

Caterpillars, radius-2 stars, and degree-3 spiders (with three legs) form the class of unlabeled level planar (ULP) trees [7], which has two known practical applications. First, the question posed in [3] of whether a path and a tree always have a simultaneous embedding with mapping remains open. Even if this cannot always be done, the question would become for which trees this is possible. The class of ULP trees partially addresses this more general question in that these trees are the only ones that can always be simultaneously embedded with any monotone path. Second, the ULP trees were used in showing that the set of minimum level non-planar (MLNP) patterns of Healy et al. [15] is incomplete [11].

2.1 Caterpillars on Three Colors

Recall that *caterpillars* are trees in which the removal of degree-1 vertices leaves in a path. Here we show that there exists a universal pointset for 3-colored caterpillars.

Theorem 1. *There exists a universal pointset P of size n on which any number of n-vertex 3-colored caterpillars can be simultaneously embedded.*

Proof. Let T be any 3-colored caterpillar on colors c_1, c_2, and c_3, where $|c_1|+|c_2|+|c_3| = n$. Let ℓ_1, ℓ_2 and ℓ_3 be three line segments each with endpoint O at the origin meeting at 120° angles. Start by placing $|c_i|$ points along ℓ_i so that the first point of each color c_i is at a distance of 1 from O, the last point is at a distance of 3 from O, and the remaining points are uniformly distributed in between. Perturb the points of each ℓ_i in a clockwise direction so that they lie along a common circular arc with no point perturbed by more than a distance ε. For sufficiently small ε, each point has a line of sight to any other point without intersecting any circular arc; see Fig 2(a).

Let S be the spine (the path after all leaves are removed) of the caterpillar T. Starting from an endpoint of S, draw all incident legs before drawing the next edge of the spine. We repeat this process for each spine vertex and its incident legs until the whole caterpillar is drawn. In doing so, pick the point of the corresponding color that is closest to the origin not already taken; see Fig 2(b). Since every point has a line of sight to any other

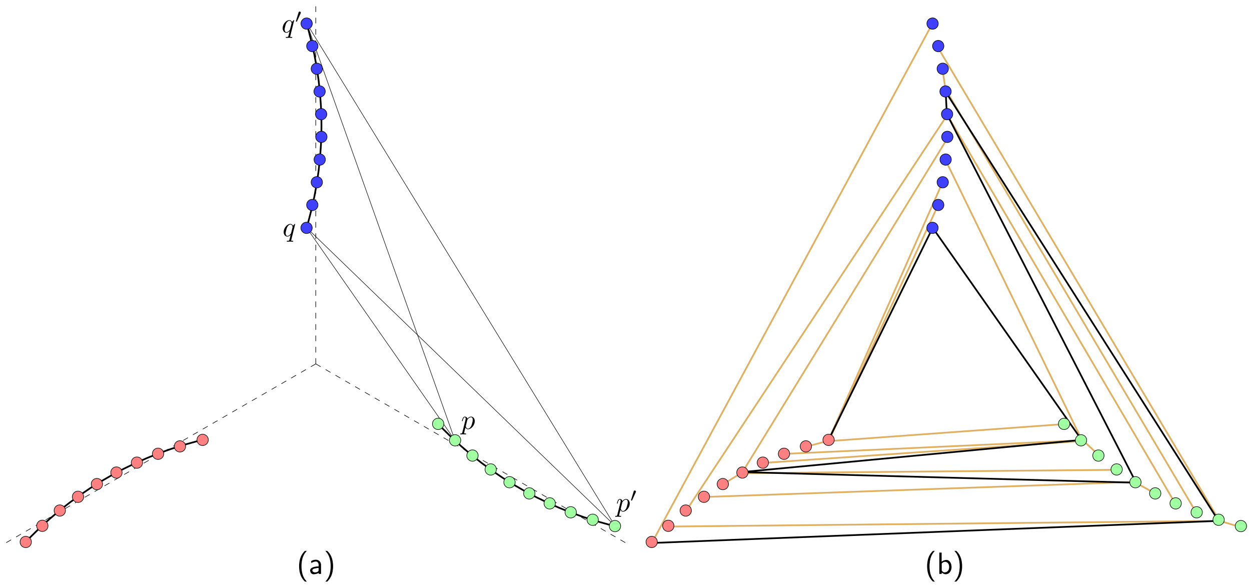

Fig. 2: The universal pointset for 3-colored caterpillars in (a) has the property where the line segment $\overline{pq}$ for any pair of points p and q does not intersect any of the circular arcs on which the points lie. This allows any caterpillar to be drawn without crossings as in (b).

point and for a given p of T, the previously drawn edges only blocks line of sight to the points already taken, a plane drawing is the result. □

2.2 Radius-2 Stars on Three Colors

A *radius-2 star* is the result of subdividing each edge of a star, a $K_{1,k}$, at most once. It is a tree consisting of any number of paths, or *legs*, of length at most 2 sharing a common endpoint r, the *root*. We show how to simultaneously embed a radius-2 star on $n+3$ points. If we constrain each color class to have the same size, $n/3$, and do not permit legs of length 1, we reduce the pointset to $n+1$ points.

Theorem 2. *There exists a universal pointset P of size $n+3$ on which any number of n-vertex 3-colored radius-2 stars can be simultaneously embedded.*

Proof. Let T be any 3-colored radius-2 star on colors c_1, c_2, and c_3, where $|c_1|+|c_2|+|c_3| = n$. Place one point p_i of each color c_i at $(i-2,0)$ so that p_2 lies at the origin O, p_1 lies one unit left of O, and p_3 lies one unit to the right of O. Let A be a concave circular arc centered above O that is visible in its entirety from each point p_i. Place $|c_1|$ points of color c_1 along the leftmost part of the arc A, followed by $|c_2|$ points of color c_2 along the central part of A, and then $|c_3|$ points of color c_3 along the rightmost part of A; see Fig. 3.

We contend that these $n+3$ points comprise a universal pointset for T. Place the root r of T (the vertex of maximum degree) on the corresponding point p_i of its color c_i, and call this point X. Each leg of radius-2 star

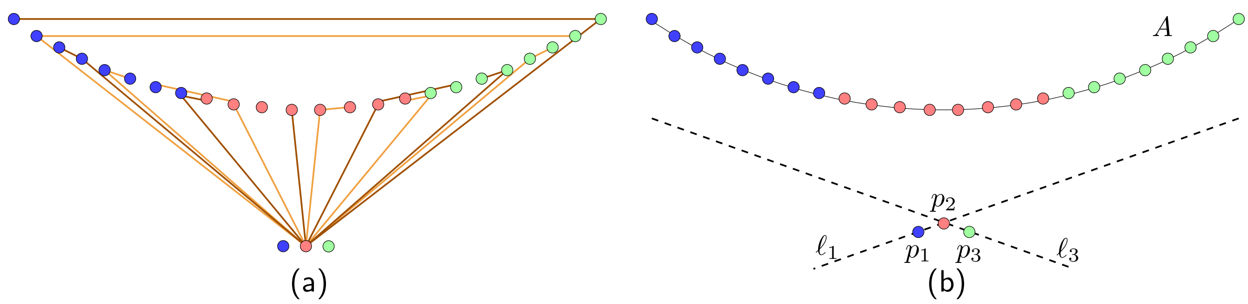

Fig. 3: The universal pointset for 3-colored radius-2 stars (a) has the property shown in (b) that all the points along the arc A are contained in the half-planes given by the lines, ℓ_1 and ℓ_3.

will be drawn from X to a point Y along the arc A with the line segment $\overline{XY}$, and if the leg has length 2, from Y to Z along A with the additional line segment $\overline{YZ}$. We need to ensure that the appropriate points Y and Z are selected for each leg of length 2 in order to avoid crossings.

We denote the legs of T of length 1 as $r{-}s$ and the legs of length 2 as $r{-}s{-}t$ where vertices s and t have colors c_j and c_k, respectively. We start by drawing each leg $r{-}s$ of length 1 in which we pick the free point nearest to X of the color c_j to be the point Y. We next draw the legs $r{-}s{-}t$ in which c_j and c_k differ. If neither c_j nor c_k is c_2, we pick the free points furthest from X of colors c_j and c_k, respectively, to be the points Y and Z. In this case, the triangle $\triangle XYZ$ contains all remaining points.

Otherwise, either c_j or c_k is c_2. We pick the free points that consist of the furthermost free point of color c_2 and the nearest free point of the other color c_1 or c_3, the colors c_j and c_k, respectively, to be the points Y and Z. In this case, the triangle $\triangle XYZ$ only contains previously used points of colors c_j and c_k. Finally, we draw the legs $r{-}s{-}t$ in which s and t share the same color c_j. We pick the remaining two points nearest to X of color c_j to be the points Y and Z. Given that any points between Y and Z along A must also be of color c_j, the triangle $\triangle XYZ$ contains no unused points.

For each leg, all remaining free points either lie inside or outside of the respective triangle $\triangle XYZ$. Hence, the line segment $\overline{YZ}$ cannot cross any previous line segment, giving a plane drawing of T; see Fig. 3. □

For some colorings of a radius-2 star, we can reduce the size of the pointset by always using the points p_1 and p_3. For instance, if the root has color c_2 so that it uses p_2, then any leg with all three colors can be drawn using p_1 and p_3 if they are placed below p_2. Alternately, if there is a leg in which the two non-root vertices have the color c_1 (or c_3), then the leg can be drawn using p_1 (or p_3) and the uppermost point of color c_1 (or c_3). In the next lemma, we show that such colorings always exist for some 3-colored radius-2 stars.

Lemma 3. *For any n-vertex radius-2 stars on colors c_1, c_2, and c_3, in which $|c_1| = |c_2| = |c_3| = n/3$, the root has color c_j, and each leg has length 2, then either (1) there is a leg with all three colors or (2) there are two legs colored $c_j{-}c_k{-}c_k$ for each $k \neq j$.*

Proof. Not all legs can be colored $c_j{-}c_j{-}c_k$ or $c_j{-}c_k{-}c_j$. This would imply $|c_j| > n/2$, contradicting $|c_j| = n/3$. Hence, at least one leg is colored $c_j{-}c_i{-}c_k$ or $c_j{-}c_k{-}c_i$ in which $i \neq j \neq k$. If $i \neq k$, then the leg has all three colors.

Otherwise, if no such leg with three colors exist, then $i = k$ so that the leg is colored $c_j{-}c_k{-}c_k$. We need to show that there exists another leg colored $c_j{-}c_l{-}c_l$ where $j \neq l \neq k$. For this not to be the case, then the only two types of legs with color c_l are colored $c_j{-}c_j{-}c_l$ or $c_j{-}c_l{-}c_j$. However, this is impossible since $|c_j| = |c_l|$. □

This lemma allows us to eliminate two of the points with colors c_1 and c_3 to give the next theorem.

Theorem 4. *There exists a universal pointset P of size $n+1$ on which any number of n-vertex 3-colored radius-2 stars can be simultaneously embedded provided that each color class has the size $n/3$ and each leg has length 2.*

Proof. We modify the pointset from Theorem 2 by moving the points p_1 of color c_1 and p_3 of color c_3 down by a distance ε to lie at $(-1, -\varepsilon)$ and $(1, -\varepsilon)$, respectively. All three points p_i still have full visibility of the circular arc A. All the points of A must lie in the intersection of the two half-planes given by the two lines ℓ_1 defined by $\overline{p_1p_2}$ and ℓ_3 defined by $\overline{p_2p_3}$; see Fig. 3(b). Furthermore, we delete one point of color c_1 and one point of color c_3 along A so that the pointset is of size $n+1$.

We have two distinct cases: either the root r has color c_2 or r has color c_1 or c_3. In the first case, all three points p_i can always be used when drawing T as previously observed, but in the second case, the point p_2 may not always be used, which is why we do not eliminate the extra point of color c_2.

If r has color c_2, we place r at point p_2. By Lemma 3, either (1) there is a leg with all three colors or (2) there are two legs, one colored $c_2{-}c_1{-}c_1$ and the other colored $c_2{-}c_3{-}c_3$. In case (1), we can draw the leg on the points p_i without blocking visibility from p_2 to any point on A since p_1 and p_3 lie below p_2, which avoids crossings. In case (2), we draw the first leg on the points p_2, p_1, and the leftmost point of color c_1 on A; then we draw the second leg on the points p_2, p_3, and the rightmost point of color c_3 on A. In neither case, do these legs block the visibility of p_2 to any other point of A.

If r has color c_1, we place r at point p_1. We apply Lemma 3, to have either (1) a leg with all three colors or (2) two legs, one colored $c_1{-}c_2{-}c_2$ and the other colored $c_1{-}c_3{-}c_3$. In case (1), we can draw the leg on the three points p_i without blocking visibility from r on p_1 to any of the other points of A since A is contained in the intersection of the half-planes given by ℓ_1 and ℓ_3.

However, in case (2), we can only draw the leg with non-root vertices colored c_3 using the points p_1, p_3, and the rightmost point of A. We cannot use the point p_2 to draw the leg with non-root vertices colored c_1 without blocking visibility of p_1 to some portion of A. An analogous argument holds if r has color c_2 in not being able to always use the point p_2. However, since both points p_1 and p_3 can always be used regardless of the color of r, this gives us our desired universal pointset of size $n+1$. □

2.3 Spiders on Two Colors

A *spider* is an arbitrarily subdivided star, $K_{1,k}$. It is a tree consisting of any number of paths, or *legs*, sharing a common endpoint r, the *root*. We present a universal pointset of size n for n-vertex spiders. We first show how to embed any spider on a larger pointset of size $n+2$ and then show how to eliminate the two extra vertices.

Lemma 5. *There exists a universal pointset P of size $n+2$ on which any number of n-vertex 2-colored spiders can be simultaneously embedded.*

Proof. Let T be any 2-colored spider on colors c_1 and c_2, where $|c_1|+|c_2|=n$. Let ℓ_1 and ℓ_2 be two rays with a common endpoint O at the origin meeting at a 90° angle; see Fig. 4(a). Start by placing $|c_i|$ points along ℓ_i so that the first point of each color ℓ_i is at the nearest integer grid position (i.e., $(1,1)$ and $(-1,1)$). The remaining points are consecutive integer grid points along the lines ℓ_i. Next we place one point p_i of each color c_i directly below the origin at positions $(0,-i)$, i.e., $(0,-1)$ and $(0,-2)$.

We show that this pointset is universal for T. Place the root r on the point of the correct color below the origin. For each leg (a 2-colored path from r to an endpoint), place its vertices at the correctly colored free point nearest to the origin O. The first path is drawn without crossings, as consecutive points along the path are either of the same color in which case there clearly are no crossings, or are of different color which leads to the path zig-zagging between points on ℓ_1 and ℓ_2. In the latter case, there are no crossings as the path always takes vertices farther away from the origin. For the k^{th} leg in the spider, the previous paths have taken a set of consecutive points of each color, defining a triangle $\triangle_{k-1}$, determined by r and the last two taken points, farthest from the origin along l_1 and l_2. The k^{th} leg is embedded as before, by placing its vertices at the correctly colored free point nearest to the origin O. As before, the edges of the k^{th} leg do not cross each other as the path zig-zags farther and farther away from O. The path does not enter the triangle $\triangle_{k-1}$ as all but the first edge of the k^{th} leg are above $\triangle_{k-1}$, and the first edge stays clear of $\triangle_{k-1}$ as it goes from r to a point either strictly to the left or to the right of $\triangle_{k-1}$. □

Next, we consider the two colorings of the root of the spider to show that we can always use the other point below the origin, thereby reducing the pointset to size n.

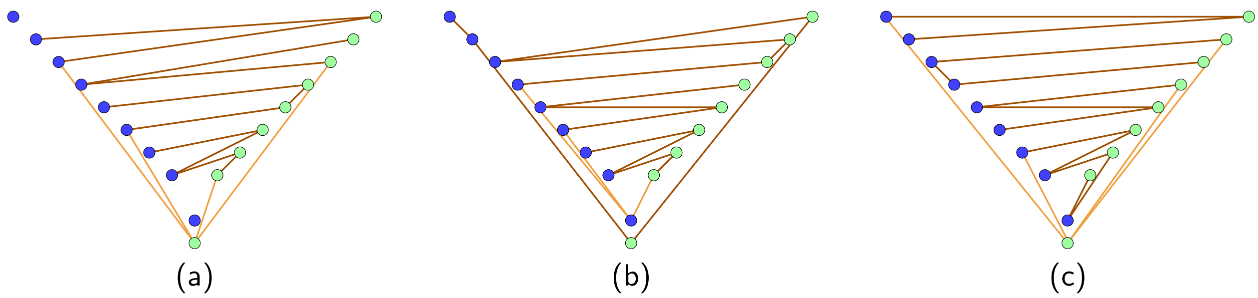

Fig. 4: The universal pointset in (a) for 2-colored spiders starts with size $n+2$ and can have size n as in (b) and (c) in which the roots of the spiders are colored c_1 and c_2, respectively.

Theorem 6. *There exists a universal pointset P of size n on which any number of n-vertex* 2*-colored spiders can be simultaneously embedded.*

Proof. We start with the pointset given in Lemma 5 and instead of placing $|c_i|$ points along ℓ_i, we only place $|c_i|-1$ points. The resulting set of points is of the desired size n and, as we show below, is universal for 2-colored spiders.

First, we consider spiders in which the root, r, has color c_1. As before, we place r on the point p_1 at $(0,-1)$. We need to show that it always possible to use the point p_2 at $(0,-2)$ of color c_2 for some leg of the spider. We use the next available correctly colored point that is nearest to O with one notable exception: We ignore the point p_2 below r until we encounter the last vertex of color c_2 in drawing T in which case we use p_2; see Fig. 4(b). This prevents the leg from self-intersecting when drawn—any edges incident to p_2 must lie along the convex hull of the points used to draw the leg.

Next, we consider spiders in which the root is colored c_2. We place r on the point p_2 at $(0,-2)$. We attempt to follow the previous strategy from Lemma 5. We draw each leg as before starting with the next available point that is nearest to O of the correct color, but this time we use the point p_1 as the first free point for the color c_1. However, this may not always work if the vertices of the first leg, $r{-}a{-}b{-}c{-}\cdots$, strictly alternate between the colors c_1 and c_2 in which a and c have color c_1 and b has color c_2. In this case, edge (r,a) crosses edge (b,c). To avoid this, we revise our strategy by reversing the order in which we draw each leg. We start with the next available correctly colored point that is the *furthest* from O, instead of the one nearest to O; see Fig. 4(c). Point p_1 is the point of color c_1 nearest to O, and hence, p_1 will be chosen as the last point for color c_1 instead of being chosen first.

The $(k+1)^{\text{th}}$ leg is fully contained inside the triangle $\triangle_k$ formed by r and the two taken points along ℓ_1 and ℓ_2, nearest to the origin O, along the k^{th} leg. These triangles nest as in Lemma 5, so no crossings are introduced between legs. Since we avoid crossings as we draw each leg, a plane drawing is the result. □

3 Universal Pointsets for Outerplanar Graphs

We extend the previous pointsets to accommodate any outerplanar graph which has as a spanning tree one of the trees from the previous section (caterpillars, radius-2 stars, and degree-3 spiders).

3.1 K_3-Caterpillars on Three Colors

A *K_3-caterpillar* is an extension of a caterpillar in which each cut-edge (u,v) can have extra incident edges (u,w) and (w,v). The resulting graph is no longer a tree given the introduction of cycles, but is an outerplanar graph with a similar structure. We take advantage of this structural similarity to show that the universal pointset for 3-colored caterpillars is also a universal pointset for 3-colored K_3-caterpillars.

Theorem 7. *There exists a universal pointset P of size n on which any number of n-vertex* 3*-colored K_3-caterpillars can be simultaneously embedded.*

Proof. We use the same pointset as in Theorem 1. We proceed to draw the K_3-caterpillar as before in which we draw legs before cut edges that are incident to the same cut vertex. However, we need to be careful when drawing the additional edges of each K_3 so as to avoid crossings. In drawing each K_3 that consists of the cut-edge (u,v) and the edges (u,w), and (w,v) (vertex w is of degree 2), we first draw the edges in the order (u,w), (w,v), and (u,v), before drawing any other edges incident to u, using the first available point of the correct color. This ensures that the edge (u,v) does not cross any other edge since the point v is still visible from u; see Fig. 5(a). Hence, a plane drawing results. □

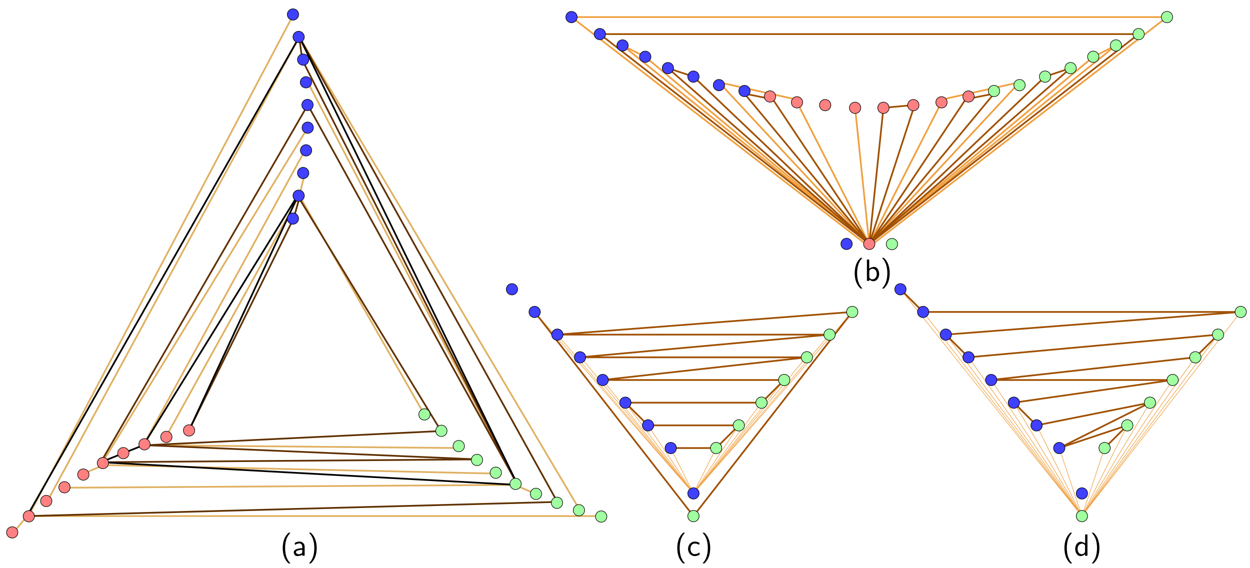

Fig. 5: Universal pointsets of size n for 3-colored K_3-caterpillar in (a), of size $n+3$ for 3-colored K_3-stars in (b), and of size $n+1$ for 2-colored fans in (c) and (d).

3.2 K_3-Stars on Three Colors

A K_3-*star* is extension of a star with root r with extra edges added between any pair of leaves s and t. The root remains the only vertex with degree greater than 2. In Theorems 2 and 4, each leg of a radius-2 star consists of two of the three edges of a triangle formed by the points r, s, and t. At any one point, the leg $r-t-s$ could be substituted for the leg $r-s-t$ so that a K_3 could be drawn on the points r, s, and t without introducing crossings; see Fig. 5(b). This gives the following two theorems:

Theorem 8. *There exists a universal pointset P of size $n+3$ on which any number of n-vertex 3-colored K_3-stars can be simultaneously embedded.*

Theorem 9. *There exists a universal pointset P of size $n+1$ on which any number of n-vertex 3-colored K_3-stars can be simultaneously embedded provided that each color class has the size $n/3$ and no vertices have degree 1.*

3.3 Fans on Two Colors

While a spider is formed by connecting an arbitrary number of paths to a single root, a *fan* is formed by connecting all vertices of a path of length $n-1$ to a single root vertex.

However, unlike the previous results of this section, the size of the universal pointset for the graph is not the same as the size of the universal pointset of its spanning tree.

Theorem 10. *There exists a universal pointset P of size $n+1$ on which any number of n-vertex 2-colored fans can be simultaneously embedded.*

Proof. We use the same pointset used in Theorem 6 except that we need an additional point of color c_1 along ℓ_1. We proceed to draw the path $s \rightsquigarrow t$ along the outerface, not including the root r, using a strategy based upon Theorem 6. We pick the nearest correctly colored free point first with two exceptions: (1) we ignore p_1 if the root has color c_2 in order to avoid a potential crossing (if the coloring of the outerface is strictly alternating) and (2) we only use p_2 for the last vertex of color c_2 if the root has color c_1. As a result, p_2 is always used, but either p_1 or the point furthest from O of color c_1 is not used. In both cases, the path $s \rightsquigarrow t$ is drawn without a self-intersection, and all points along the path are visible from point of r. Hence, an edge can then be added from r to each of the other $n-1$ points in the pointset on which $s \rightsquigarrow t$ was drawn. This clearly yields a plane drawing. □

4 More Graphs Without Simultaneous Embeddings

We next consider graphs without simultaneous embeddings for $2 \leq k \leq 5$ colors.

4.1 Two Colors

Fowler et al. [9] presented a set of 16 pairs of planar graphs that cannot always be simultaneously embedded whose unions are either homeomorphic to K_5 or to $K_{3,3}$. Most of these pairs require for the vertices to be distinctly colored. However, the $5-$vertex pair given in Fig. 6(a) can be 2-colored and still not always have a simultaneous embedding as we show next.

Theorem 11. *There exist a* 2*-colored planar graph and a pseudo-forest that cannot be simultaneously embedded.*

Proof. Let G_1 be the planar graph consisting of a K_5 minus an edge e in which the endpoints of the missing edge are red and the remaining three vertices are blue. Let G_2 be the pseudo-forest consisting of a K_3 on three blue vertices and an non-incident edge on two red vertices; see Fig. 6(a). In both G_1 and G_2, the vertices with the same color have the same adjacencies. Hence, there is only one non-isomorphic mapping of the vertices onto the three blue points and the two red points. Each pair of non-incident edges in the non-planar K_5 union is present in one of the graphs. In any straight-line embedding of the vertices of these graphs in the plane, at least one of the graphs must have a crossing as incident straight-line edges cannot cross. □

4.2 Three Colors

We next show that there exist outerplanar triples on 3 colors that cannot be simultaneously embedded.

Theorem 12. *There exists three* 3*-colored outerplanar graphs that cannot be simultaneously embedded.*

Proof. Consider the following three 3-colored outerplanar graphs (the third is a path) on 5 points given in Fig. 6(b) whose union is K_5:

1. $a-b-c-d-e-a$ and $b-e$ (dark blue edges),
2. $a-c-e-b-d-a$ and $b-a-e$ (light yellow edges),
3. $b-c-a-d-e$ (dashed red edges),

where a is yellow, b and e are blue, and c and d are red.

For each graph, vertices with the same color have the same adjacencies. Hence, there is only one non-isomorphic mapping of the vertices onto one yellow point, two blue points and two red points. Each pair of non-incident edges in the non-planar union is in one of the three graphs, forcing at least one of the graphs to have a crossing. □

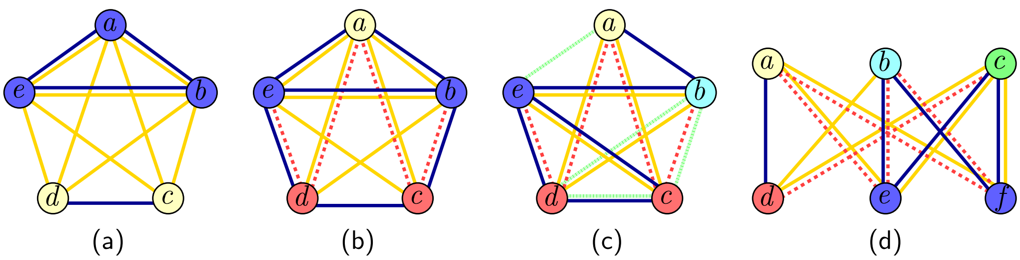

Fig. 6: A planar graph and a pseudo-forest on 2 colors in (a), three outerplanar graphs on 3 colors in (b), four pseudo-forests on 4 colors in (c), and three pseudo-forests on 5 colors in (d) without simultaneous embeddings.

4.3 Four and Five Colors

We next turn our attention to pseudo-forests, a proper subclass of outerplanar graphs.

Theorem 13. *There exists four* 4*-colored pseudo-forests and three* 5*-colored pseudo-forests that cannot be simultaneously embedded.*

Proof. Consider the following four 4-colored pseudo-forests shown in Fig. 6(c) whose union is K_5:

1. $a-b-e-c-d-e$ (dark blue edges),
2. $a-c-e-b-d-a$ (light yellow edges),
3. $b-c-a-d-e$ (dashed red edges),
4. $b-c-d-b$ and $a-e$ (dash-3 dot green edges),

where only c and d have the same color and the following three 5-colored pseudo-forests shown in Fig. 6(d) whose union is $K_{3,3}$:

1. $a-d$ and $b-f-c-e-b$ (dark blue edges),
2. $a-e-c-f-a$ and $c-d-b$ (light yellow edges),
3. $a-e-b-f-a$ and $c-d$ (dashed red edges),

where only e and f share a color. For each graph, vertices with the same color have the same adjacencies, so there is only one non-isomorphic mapping of the vertices onto points of the same color. Each pair of non-incident edges in the non-planar union is in one of the graphs, forcing at least one graph to have a crossing. □

5 Conclusions and Open Problems

We introduced the problem of small sets supporting colored simultaneous embedding (CSE) and universal CSE pointsets. We provided universal or near-universal pointsets for caterpillars and radius-2 stars on 3 colors and degree-3 spiders on 2 colors, which comprise the three classes of ULP trees [7]. However, we only have partial results for ULP outerplanar graphs. Of all of the classes of ULP graphs described in [10], two subclasses are outerplanar: (1) *outerplanar generalized caterpillars*, a superset of the K_3-caterpillars for which we gave a 3-colored universal pointset, and (2) *1-connected extended degree-3 spiders*, a proper subsets of fans for which we gave a 2-colored near-universal pointset; see Fig. 7(b) and (c) in the appendix. This leaves the question of whether there exists a 3-colored universal pointset for all ULP trees, and if so, for all ULP outerplanar graphs.

In the context of CSE, previous examples of graphs without simultaneous embeddings, such as the 16 pairs given in [9], required for each vertex to be distinctly colored. We presented several examples in which this was not the case. We accomplished this by ensuring that vertices with the same color had the same adjacencies. Table 1 summarizes the current status of colored simultaneous embedding. A "✓" indicates that it is always possible to simultaneously embed the type of graphs, a "✗" indicates that it is not always possible, and a "**?**" indicates an open problem. New results are in bold with asterisks.

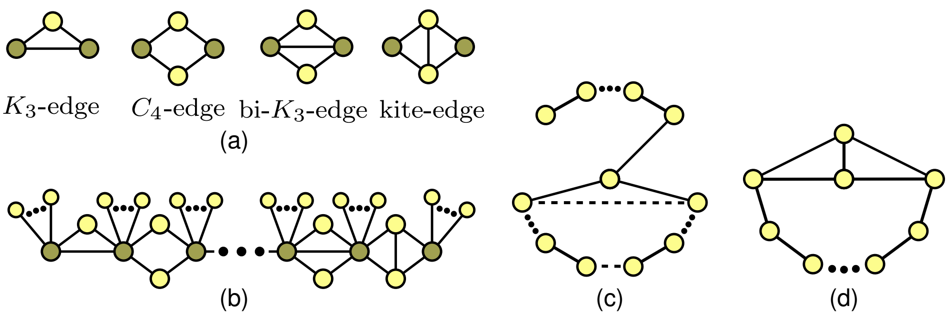

Fig. 7: Four types of outerplanar "edges" in (a) are used to construct outerplanar generalized caterpillars as in (b). Extended degree-3 spiders (every spanning tree is degree-3 spider or a path) can be 1-connected as in (c) or 2-connected as in (d).

$k =$	1	2	3	4	5	6	9	n
*** **Three paths** $P_1 \ldots P_3$	✓	✓	✓	?	?	?	✗ [3]	✗
*** **Four paths** $P_1 \ldots P_4$	✓	✓	✓	?	?	✗	✗	✗
*** **Three cycles** $C_1 \ldots C_3$	✓	✓	✓	?	?	✗	✗	✗
Two outerplanar graphs O_1, O_2 [3, 12]	✓	?	?	?	?	✗	✗	✗
*** **Three outerplanar graphs** $O_1 \ldots O_3$	✓	?	✗	✗	✗	✗	✗	✗
*** **Three pseudo-forests** $F_1 \ldots F_3$	✓	?	?	?	✗	✗	✗	✗
*** **Four pseudo-forests** $F_1 \ldots F_4$	✓	?	?	✗	✗	✗	✗	✗
*** **Any number of paths**	✓	✓	✓	?	✗	✗	✗	✗
*** **Any number of caterpillars**	✓	✓	✓	?	✗	✗	✗	✗
Two trees T_1, T_2 [16]	✓	?	?	?	?	?	?	✗
*** **Tree** T **and path** P	✓	✓	?	?	?	?	?	?
*** **Outerplanar graph** G **and path** P	✓	✓	?	?	?	?	?	?
Planar graph G and path P [3, 6]	✓	?	?	?	?	?	✗	✗
*** **Planar graph** G **and pseudo-forest** F	?	✗	✗	✗	✗	✗	✗	✗

Table 1: k-colored simultaneous embeddings on n points: new results and open problems.

References

1. P. Bose. On embedding an outer-planar graph in a point set. *Computational Geometry: Theory and Applications*, 23(3):303–312, 2002.
2. U. Brandes, C. Erten, J.J. Fowler, F. Frati, M. Geyer, C. Gutwenger, S. Hong, M. Kaufmann, S.G. Kobourov, G. Liotta, P. Mutzel, and A. Symvonis. Colored simultaneous geometric embeddings. In *13th Computing and Combinatorics Conference*, volume 4598 of *LNCS*, pages 254–263, 2007.
3. P. Brass, E. Cenek, C. A. Duncan, A. Efrat, C. Erten, D. Ismailescu, S. G. Kobourov, A. Lubiw, and J. S. B. Mitchell. On simultaneous graph embedding. *Computational Geometry: Theory and Applications*, 36(2):117–130, 2007.
4. H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990.
5. Bradley Dux, Anand Iyer, Saumya Debray, David Forrester, and Stephen G. Kobourov. Visualizing the behavior of dynamically modifiable code. In *13th Workshop on Program Comprehension*, pages 337–340, 2005.
6. C. Erten and S. G. Kobourov. Simultaneous embedding of planar graphs with few bends. *Journal of Graph Algorithms and Applications*, 9(3):347–364, 2005.
7. A. Estrella-Balderrama, J. J. Fowler, and S. G. Kobourov. Characterization of unlabeled level planar trees. *Computational Geometry: Theory and Applications*. To appear in 2009.
8. A. Estrella-Balderrama, J. J. Fowler, and S. G. Kobourov. Characterization of unlabeled level planar trees. In *14th Symposium on Graph Drawing*, volume 4372 of *LNCS*, pages 367–369, 2007.
9. J. J. Fowler, M. Jünger, S. G. Kobourov, and M. Schulz. Characterizations of restricted pairs of planar graphs allowing simultaneous embedding with fixed edges. In *34th International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2008)*. To appear in 2008.
10. J. J. Fowler and S. G. Kobourov. Characterization of unlabeled planar graphs. In *15th Symposium on Graph Drawing (GD 2007)*, volume 4875 of *LNCS*, pages 37–49, 2008.
11. J. J. Fowler and S. G. Kobourov. Minimum level nonplanar patterns for trees. In *15th Symposium on Graph Drawing (GD 2007)*, volume 4875 of *LNCS*, pages 69–75. Springer, 2008.
12. F. Frati. Embedding graphs simultaneously with fixed edges. In *14th Symposium on Graph Drawing (GD 2006)*, volume 4372 of *LNCS*, pages 108–113. Springer, 2007.
13. F. Frati, M. Kaufmann, and S. G. Kobourov. Constrained simultaneous and near-simultaneous embeddings. In *15th Symposium on Graph Drawing*, volume 4875 of *LNCS*, pages 268–279, 2008.
14. P. Gritzmann, B. Mohar, J. Pach, and R. Pollack. Embedding a planar triangulation with vertices at specified points. *American Math. Monthly*, 98:165–166, 1991.
15. Patrick Healy, Ago Kuusik, and Sebastian Leipert. Characterization of level non-planar graphs by minimal patterns. In *6th Computing and Combinatorics Conference*, pages 74–84. 2000.
16. M. Kaufmann, I. Vrťo, and M. Geyer. Two trees which are self-intersecting when drawn simultaneously. In *13th Symposium on Graph Drawing*, volume 3843 of *LNCS*, pages 201–210, 2006.
17. P. Rosenstiehl and R. E. Tarjan. Rectilinear planar layouts and bipolar orientations of planar graphs. *Discrete Computational Geometry*, 1(4):343–353, 1986.